BOUNDARY CONTROL OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

BY

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To my Parents and Family

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THESIS ABSTRACT

Name:	Faez Ali Nasser AL-Qarni
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The objective in this work is to find a stabilizing control for the one-dimensional linear reaction-advection-diffusion PDE system with space and time dependent coefficients using a transmutation operator. This method consists in finding an invertible transformation mapping the given system to a new one whose stability has been proved. A few numerical examples are presented to illustrate the usefullness of the approach.

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ملخص الرساله

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الموضوع في هذا العمل هو ايجاد السيطره المستقره لمعادله انتشار الحرارة عندما يكون معامل الانتشار معتمد علي الزمن والبعد باستخدام تحويل تفاضلي خطي .

هذا الطريقة تعتمد علي ايجاد تحويل تفاضلي خطي قابل للانعكاس ثم تحويل النظام الاصلي عبر نظام مستقر مثبت سابقا.

في الاخير تم عرض امثله بسيطه للاستفاده من هذه الطريقة.

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Chapter 1

INTRODUCTION

1.1 Introduction

Many physical systems and processes are modeled by partial differential equations (PDE). The control of such infinite dimensional systems has been an active area of research since at least 1960s and this period became the most important period for the development of control theory and adaptive control. In two settings control of PDEs comes depending on where the sensors and actuators are located. In domain control, where the actuation penetrates inside part of the domain of the PDE system or is distributed everywhere in the domain and control, where the actuation and sensing are applied only through the boundary conditions. Because actuation and sensing are nonintrusive, boundary control is generally considered to be physically feasible and easier to implement. However boundary control is a much harder problem. Only a few methods have been developed over the years for boundary control problems of PDEs and most of the studies either do not cover boundary control or dedicate only small fractions of their coverage to boundary control. Some of the initial efforts in the 1960s and early 1970s were addressing the controllability and optimal control of linear systems. Adaptive control has been the subject of active research for over three decades now. There have been many theoretical successes, including the development of rigorous proofs of stability and an understanding of the dynamical properties of adaptive schemes. Several successful applications have been reported and the last ten years have seen an impressive growth in the availability of commercial adaptive controllers. The area of adaptive control has grown to be one of the richest in terms of algorithms, design techniques and analytical tools. Several books and research monographs already exist on the topics of parameter estimation and adaptive control. The advances in stability theory and the progress in control theory in the 1960s improved the understanding of adaptive control and contributed to a strong renewed interest in the field in the 1970s. Around 2000, Krstic and Smyshlyaev initiated an effort to extend backstepping to partial differential equation (PDEs) in the context of boundary control. State space techniques and stability theory based on Lyapunov theory were introduced. Backstepping is unlike any of the methods previously developed for control of PDEs. Appropriate references in this area are the books by J. L. Lions [23], Curtain and Zwart [7], Lasiecka and Triggiani [22], Bensoussan et al. [3], and Christofides [6]. Application-oriented books on control of PDEs have been dedicated to problems that arise from flexible structures [26], [20], [21], [2], [12] and from flow control [1], [11].

In view of the curse of dimensionality which arises when one tries to numerically solve such problems, a new method has been introduced to tackle such problems: the backstepping method. It is an adaptation to the PDE case of a method initially used for ODE systems.

Some of the advantages of the method are

- its simplicity
- ease of implementation by considering boundary control of PDE and the use of an invertible transformation mapping the given system to a stable system.

As for the disadvantages, we may list the facts that

- it is not necessarily optimal in any sense.

- one has to find the appropriate transformation for each given system.

My work is devoted to the boundary control of systems governed by parabolic PDEs.

1.2 Literature overview

Early works on adaptive control of infinite-dimensional systems, surveyed by Logemann and Townley [25], were for plants stabilizable by non-identifier based high gain feedback. Model Reference Adaptive Control (MRAC) type schemes were designed by Hong and Bentsman [13], Bohm, Demetriou, Reich, and Rosen [5], Solo and Bamieh [31], Orlov [27], and Bentsman and Orlov [4] and others. While the strength of these results are the proofs of identifiability of infinite dimensional parameter vectors, their limitation is that they require control action throughout the PDE domain. Other efforts such as of Demetriou and Ito [9] and Wen and Balas [33] have employed tools from positive realness; they have also provided some cunning examples that go beyond the relative degree one restriction. Adaptive linear quadratic control with least-squares estimation was pursued by Duncan, Maslowski, and Pasik-Duncan [10] for linear stochastic evolution equations with unbounded input operators and exponentially stable dynamics. Adaptive control of nonlinear PDEs has also received some attention. Liu and Krstic [24] and Kobayashi [17] considered a Burgers equation with various parametric uncertainties. Jovanovic and Bamieh [14] designed adaptive controllers for nonlinear systems on lattices. An experimentally validated adaptive boundary controller for a flexible beam was presented by de Queiroz, Dawson, Agarwal, and Zhang [8]. In

2006, Jia [15] studied the control of the linear heat equation with a space and time dependent coefficient function by Dirichlet and Neumann boundary control laws. This equation models the heat diffusion and space, time dependent heat generation in a one dimensional rod. Without control, the system is unstable if the coefficient function is positive and large. With boundary control based on the state feedback, he has shown that for the time analytic coefficient $\lambda(x,t)$ diffusion term, the exponential stability of the system at any rate can be achieved at the expence of having a time dependent kernel. In 2004, Smyshlyaev and Krstic [30] introduced adaptive controllers by designing parameter identifiers and substituting the parameter estimates they generate into the control law, and in 2007, [28], they studied the boundary control problem for a class of unstable 3D reaction-advection-diffusion PDEs with unknown coefficients. For their problem (even in 1D), due to the absence of parametrized families of controllers, no solution exists. They proved the "separation principle"—the global stability of such a nonlinear closed-loop PDE system. Also at that time, they developed output-feedback adaptive controllers for two benchmark parabolic PDEs motivated by a model of thermal instability in solid propellant rockets. Both benchmark plants are unstable, have infinite relative degree, and are controlled from the boundary. One plant has an unknown parameter in the PDE and the other in the boundary condition. In 2008, Krstic and Smyshlyaev [19] developed adaptive controllers for parabolic partial differential equations (PDEs) controlled from the boundary and containing unknown destabilizing parameters affecting the interior of the domain. These were the first adaptive controllers for unstable PDEs without relative degree limitations, open-loop stability assumptions, or domain-wide actuation. It was the first necessary step towards developing adaptive controllers for physical systems such as fluid, thermal, and chemical dynamics, where actuation can only be applied non-intrusively, the dynamics are unstable, and the parameter, such as the Reynolds, Rayleigh, Prandtl, or Peclet numbers are unknown because they vary with the operating conditions. Their method is built upon their explicitly parametrized control formulae in [30] to avoid solving Riccati or Bezout equations at each time step.

1.3 Objectives

Our objective in this work is to revisit the backstepping technique as presented in the works of kristic and co-workers by introducing the idea of a transmutation operator well known in spectral and scattering theories. Then, tackle the following two problems:

1. Find a stabilizing control for the one-dimensional linear reaction-advection-diffusion PDE system with space dependent coefficients and separated boundary conditions

$$\begin{cases} u_t(x,t) = a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t) \quad 0 < x < 1,\ t > 0 \\ \alpha u(0,t) + \beta u_x(0,t) = 0,\ t > 0 \\ \gamma u(1,t) + \delta u_x(1,t) = U(t),\ t > 0 \\ u(x,0) = \phi(x),\ t > 0 \end{cases} \tag{1.1}$$

   where U is the boundary control.

2. Find a stabilizing control when the coefficients of the PDE are space and time dependent with expanded boundary conditions, that is

$$\begin{cases} u_t(x,t) = a(x,t)u_{xx}(x,t) + b(x,t)u_x(x,t) + c(x,t)u(x,t), \quad 0 < x < 1,\ t > 0 \\ \alpha u(0,t) + \beta u_x(0,t) = 0,\ t > 0 \\ \gamma u(1,t) + \delta u_x(1,t) = U(t),\ t > 0 \\ u(x,t) = \phi(x),\ t > 0 \end{cases} \tag{1.2}$$

where U is the boundary control, using transmutation operators for both problems.

In the above problems the coefficient functions belong to appropriate spaces

1.4 Methodology

We transform first the system (1.1) into

$$\begin{cases} v_\tau(z,\tau) = v_{zz}(z,\tau) + q(z)v(z,\tau), \ 0 < z < 1, \ \tau > 0 \\ \alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0, \ \tau > 0 \\ \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) = V(\tau), \ \tau > 0 \\ v(z,0) = \psi(\tau)., \ 0 < z < 1 \end{cases} \tag{1.3}$$

where V is the boundary control, using the change of variables

$$\begin{cases} t = \frac{\tau}{a_0} \\ z = \sqrt{a_0} \int_0^x \frac{ds}{\sqrt{a(s)}} \\ v(z,\tau) = a^{-\frac{1}{4}}(x)u(x,t)\exp(\int_0^x \frac{b(s)}{2a(s)} ds) \end{cases} \tag{1.4}$$

where

$$a_0 = \left(\int_0^1 \frac{ds}{\sqrt{a(s)}} \right)^{-2}$$

then a transmutation operator is used to map the resulting system into the stable target system

$$\begin{cases} \hat{v}_\tau(z,\tau) = \hat{v}_{zz}(z,\tau) + \lambda \hat{v}(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 \hat{v}(0,\tau) + \beta_1 \hat{v}_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 \hat{v}(1,\tau) + \delta_1 \hat{v}_z(1,\tau) = 0,\ \tau > 0 \\ \hat{v}(z,0) = \hat{\psi}(z),\ 0 < z < 1 \end{cases} \tag{1.5}$$

whose stability will be assessed where $\hat{\psi}$ is a continuous function.

Next, we consider the case where the coefficients are space and time dependent, and assume that the system (1.2) has already been transformed to

$$\begin{cases} v_\tau(z,\tau) = v_{zz}(z,\tau) + q(z,\tau)v(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 v(1,\tau) + \delta_1 v_z(1,t) = V(\tau),\ \tau > 0 \\ v(z,0) = \psi(z),\ 0 < z < 1 \end{cases} \tag{1.6}$$

where V is the boundary control. In this case we write $q(z,\tau) = q_0(z) + q_1(z,\tau)$, (we may as well take $q_0(z) = q(z,0)$) and map the system

$$\begin{cases} v_\tau(z,\tau) = v_{zz}(z,\tau) + q_0(z)v(z,\tau) + q_1(z,\tau)v(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) = V(\tau),\ \tau > 0 \\ v(z,0) = \psi(z),\ 0 < z < 1 \end{cases}$$

into the target system

$$\begin{cases} \hat{v}_\tau(z,\tau) = \hat{v}_{zz}(z,\tau) + \lambda \hat{v}(z,\tau) + \hat{H}(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 \hat{v}(0,\tau) + \beta_1 \hat{v}_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 \hat{v}(1,\tau) + \delta_1 \hat{v}_z(1,\tau) = 0,\ \tau > 0 \\ \hat{v}(z,0) = \hat{\psi}(z),\ 0 < z < 1 \end{cases}$$

whose stability will be assessed. Where $H(z,\tau) = q_1(z,\tau)v(z,\tau)$, $\psi(z)$ and $\hat{\psi}(z)$ are continuous functions.

We shall work out a few examples to illustrate the usefulness of the approach.

This thesis is organized as follows: In chapter two, we present the basic definitions, lemmas, properties and notation needed later in this work. In chapter three, we present the backstepping method to design boundary controllers stabilizing the PDE system. These adaptive controllers work " irrespective" of the initial condition. We shall present our contribution in chapter four specifically we revisit the backstepping method and introduce the idea of transmutation well known in spectral and scattering theories and consider boundary control of parabolic systems with time and space dependent coefficients using transmutation.

Chapter 2

PRELIMINARIES

This chapter introduces the notation used in this thesis, as well as basic definitions and results.

2.1 Lyapunov Stability

2.1.1 Definition of Stability

Before we study the stability for PDEs under consideration, we mention some of the basics of stability analysis for linear ODEs. We focus only on linear PDEs in this study.

We consider systems described by ordinary differential equations of the form

$$\dot{z} = Az \tag{2.1}$$

where $z \in \mathbb{R}^n$ and A is an nxn real matrix. (2.1) is said to be exponentially stable at $z = 0$ if there exist positive constants M and α such that

$$\| z(t) \| \leq Me^{-\alpha t} \| z(0) \| \text{ for all } t \geq 0 \tag{2.2}$$

where $\| . \|$ denotes a vector norm. We can test the exponential stability by verifying that all the eigenvalues of the matrix A have negative real parts. But this test is not always practical so we can use the Lyapunov second method, which is presented next.

The system (2.1) is exponentially stable in the sense of definition (2.2) if and only if for any positive definite nxn matrix Q there exists a positive definite and symmetric matrix P such that

$$PA + A^T P = -Q \tag{2.3}$$

In this case, if

$$v(z) = z^T P z \tag{2.4}$$

then,

$$\dot{v} = \frac{dv(z(t))}{dt} = \dot{z}^T P z + z^T P \dot{z}$$

using (2.1) we get,

$$\begin{aligned} \dot{v} &= (Az)^T P z + z^T P (Az) = z^T A^T P z + z^T P A z \\ &= z^T (A^T P + PA) z = -\ z^T Q z \ < 0 \end{aligned} \tag{2.5}$$

from which we get,

$$v(z(t)) \to 0 \quad \text{as} \qquad t \to \infty$$

so that $z(t) \to 0$ as $t \to \infty$.

v is called a Lyapunov function.

2.1.2 Normalization of the Basic Parabolic PDE

The aim is to develop a basic "non-dimensionalized" PDE model, which will be the starting point for many of the analysis and control design considerations in this study.

Consider a single dimensional heat conduction model or diffusion model, which can both be represented as

$$\begin{cases} \tilde{T}_t(\xi,\tau) = \varepsilon \tilde{T}_{\xi\xi}(\xi,\tau) + \mu \tilde{T}(\xi,\tau), \ 0 < \xi < l, \ \tau > 0 \\ \tilde{T}(0,\tau) = T_1, \ \tau > 0 \\ \tilde{T}(l,\tau) = T_2, \ \tau > 0 \\ \tilde{T}(\xi,0) = \tilde{T}_0(\xi), \ 0 < \xi < l \end{cases} \tag{2.6}$$

where $\tilde{T}(\xi,\tau)$ is a function of the spatial variable ξ and time τ, representing the temperature and the initial temperature distribution is $\tilde{T}_0(\xi)$ and the ends of the rod are kept at constant temperatures T_1 and T_2, ε denotes the thermal diffusivity, and μ a positive constant.

To normalize the above system we proceed as follows:

1. We scale ξ to normalize the length $x = \frac{\xi}{l}$ to obtain,

$$\begin{cases} T_\tau(x,\tau) = \frac{\varepsilon}{l^2} T_{xx}(x,\tau) + \mu T(x,\tau), \ 0 < x < 1, \ \tau > 0 \\ T(0,\tau) = T_1, \ \tau > 0 \\ T(1,\tau) = T_2, \ \tau > 0 \\ T(x,0) = T_0(x), \ 0 < x < 1 \end{cases}$$

2. We scale time $\tau, t = \frac{\varepsilon}{l^2}\tau$ to normalize the diffusion coefficient to get,

$$\begin{cases} T_t(x,t) = T_{xx}(x,t) + \lambda T(x,t),\ 0 < x < 1,\ t > 0 \\ T(0,t) = T_1,\ t > 0 \\ T(1,t) = T_2,\ \tau > 0 \\ T(x,0) = T_0(x),\ 0 < x < 1 \end{cases}$$

where $\lambda = \frac{l^2}{\varepsilon}\mu$.

3. We introduce the new variable

$$w = T - \bar{T}$$

where

$$\bar{T}(x) = T_1 \cos\sqrt{\lambda}x + \frac{T_2 - T_1\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}\sin\sqrt{\lambda}x$$

is the solution to the Boundary value problem

$$\begin{cases} \bar{T}''(x) + \lambda\bar{T}(x) = 0 \\ \bar{T}(0) = T_1 \\ \bar{T}(1) = T_2 \end{cases} \tag{2.7}$$

so that,

$$\begin{cases} w_t(x\,,t) = w_{xx}(x\,t) + \lambda w(x,t),\; 0 < x < 1,\, t > 0 \\ w(0,t) = 0,\, t > 0 \\ w(1,t) = 0,\, t > 0 \\ w(x,0) = w_0(x),\; 0 < x < 1 \end{cases} \tag{2.8}$$

where $w_0(x) = T_0(x) - \bar{T}_0(x)$.

Remark when $\lambda \to 0$ the above formula for $\bar{T}(x)$ reduces to

$$\bar{T}(x) = T_1 + (T_2 - T_1)x.$$

Remark 1 *The following are the basic types of boundary conditions at $x = 0$ for PDEs in one dimension:*

* *Dirichlet: $w(0,t) = 0$ (fixed temperature at $x = 0$)*
* *Neumann: $w_x(0,t) = 0$ (fixed heat flux at $x = 0$)*
* *Robin: $w_x(0,t) + qw(0,t) = 0$ (mixed).*

In this work we will be studying problems with all three types of boundary conditions.

2.1.3 Stability of the PDE System

Consider the initial boundary value problem,

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + \lambda w(x,t), \ 0 < x < 1, \, t > 0 \\ w(0,t) = 0, \, t > 0 \\ w(1,t) = 0, \, t > 0 \\ w(x,0) = w_0(x), \ 0 < x < 1 \end{cases} \tag{2.9}$$

where $w_0(x)$ is a continuous function. As in finite dimension, there are two ways to analyze stability properties: find the exact solution or use Lyapunov theory to show stability without solving the PDE. Both methods can be applied in this case.

The most common Lyapunov function for PDEs in L_2 spatial norm is

$$v(t) = \frac{1}{2}\int_0^1 w^2(x,t)dx = \frac{1}{2}\left\|w(.,t)\right\|^2. \tag{2.10}$$

Taking the time derivative of v we obtain,

$$\frac{dv(t)}{dt} = \int_0^1 w(x,t)w_t(x,t)dx.$$

Using (2.9), we get,

$$\frac{dv(t)}{dt} = \int_0^1 w(x,t)[w_{xx}(x,t) + \lambda w(x,t)]dx$$

Integration by parts gives,

$$\frac{dv(t)}{dt} = w(x,t)w_x(x,t)|_{x=0}^{1} - \int_0^1 w_x^2(x,t)dx + \lambda \int_0^1 w^2(x,t)dx$$

leading to

$$\frac{dv(t)}{dt} = -\int_0^1 w_x^2(x,t)dx + \lambda \int_0^1 w^2(x,t)dx \tag{2.11}$$

We shall find an upper bound to the right-hand side of (2.11) in terms of v. For this, we shall recall the following results,

1) **Young's Inequality**

$$ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2 \tag{2.12}$$

2) **Cauchy-Schwarz Inequality**

$$\int_0^1 uwdx \leq \|u\| \|w\| \leq \frac{\alpha}{2}\|u\|^2 + \frac{1}{2\alpha}\|w\|^2 \tag{2.13}$$

and

Lemma 1 ***(Poincare Inequality)*** *For any u continuously differentiable on* $[0,1]$

$$\begin{aligned}\int_0^1 u^2dx &\leq 2u^2(1) + 4\int_0^1 u_x^2dx \\ \int_0^1 u^2dx &\leq 2u^2(0) + 4\int_0^1 u_x^2dx.\end{aligned} \tag{2.14}$$

Now, back to equation (2.11)

$$\frac{dv(t)}{dt} = -\int_0^1 w_x^2(x,t)dx + \lambda \int_0^1 w^2(x,t)dx$$

by (2.14) we have, assuming $\lambda < \frac{1}{4}$,

$$\begin{aligned}
\frac{dv(t)}{dt} &\leq -\frac{1}{4}\int_0^1 w^2(x,t)dx + \lambda \int_0^1 w^2(x,t)dx \\
&\leq -\frac{1}{4}\|w(.,t)\|^2 + \lambda \|w(.,t)\|^2 = (\lambda - \frac{1}{4})\|w(.,t)\|^2 \\
\frac{dv(t)}{dt} &\leq 2(\lambda - \frac{1}{4})v(t)
\end{aligned}$$

Therefore

$$v(t) \leq v(0)e^{2(\lambda-\frac{1}{4})t} \tag{2.15}$$

that is

$$\|w(x,t)\| \leq e^{(\lambda-\frac{1}{4})t}\|w_0(x)\|$$

Thus, as $t \to \infty$, $\|w(.,t)\| \to 0$ but this does not imply that $w(x,t) \to 0$ as $t \to \infty$, for all $x \in (0,1)$.

2.2 Pointwise Stability:

Define H_1 norm as

$$\|w\|_{H_1} := \left(\int_0^1 w^2(x,t)dx + \int_0^1 w_x^2(x,t)dx\right)^{\frac{1}{2}}. \tag{2.16}$$

We shall prove the following result,

$$\max_{x\in[0,1]} |w(x,t)| \leq ce^{(\lambda-\frac{1}{4})t} \max \|w(x,0)\|_{H_1} \tag{2.17}$$

for some $c > 0$.

To prove (2.17) we need the following result,

Lemma 2 ***(Agmon's Inequality)*** *For a function $w \in H_1$, the following inequalities hold*

$$\begin{aligned}
\max_{x\in[0,1]} |w(x,t)|^2 &\leq w(0,t)^2 + 2\|w(.,t)\| \|w_x(.,t)\|, \qquad (2.18)\\
\max_{x\in[0,1]} |w(x,t)|^2 &\leq w(1,t)^2 + 2\|w(.,t)\| \|w_x(.,t)\|.
\end{aligned}$$

We shall consider instead of v given in (2.10),

$$\begin{aligned}
v(t) &= \frac{1}{2}\int_0^1 w^2(x,t)dx + \frac{1}{2}\int_0^1 w_x^2(x,t)dx\\
&= \frac{1}{2}(\|w(.,t)\|^2 + \|w_x(.,t)\|^2).
\end{aligned}$$

We have,

$$
\begin{aligned}
\frac{dv(t)}{dt} &= \int_0^1 w(x,t)w_t(x,t)dx + \int_0^1 w_x(x,t)w_{xt}(x,t)dx \\
&= \int_0^1 w(x,t)w_{xx}(x,t)dx + \lambda\int_0^1 w^2(x,t)dx + \int_0^1 w_x(x,t)w_{xt}(x,t)dx \\
&= w(x,t)w_x(x,t)|_{x=0}^1 - \int_0^1 w_x^2(x,t)dx + \lambda\int_0^1 w^2(x,t)dx + w_x(x,t)w_t(x,t)|_{x=0}^1 \\
&\quad - \int_0^1 w_{xx}(x,t)w_t(x,t)dx \\
&= -\int_0^1 w_x^2(x,t)dx + \lambda\int_0^1 w^2(x,t)dx - \int_0^1 w_{xx}^2(x,t)dx - \lambda\int_0^1 w_{xx}(x,t)w(x,t)dx \\
&= -\int_0^1 w_x^2(x,t)dx + \lambda\int_0^1 w^2(x,t)dx - \int_0^1 w_{xx}^2(x,t)dx - \lambda[w(x,t)w_x(x,t)|_{x=0}^1 \\
&\quad - \int_0^1 w_x^2(x,t)dx] \\
\frac{dv(t)}{dt} &= -\int_0^1 w_x^2(x,t)dx - \int_0^1 w_{xx}^2(x,t)dx + \lambda(\int_0^1 w^2(x,t)dx + \int_0^1 w_x^2(x,t)dx)
\end{aligned}
$$

Using Poincare Inequality (2.14), and assuming $\lambda < \frac{1}{4}$, we obtain,

$$\frac{dv(t)}{dt} \le -\frac{1}{4}(\|w\|^2 + \|w_x\|^2) + \lambda(\|w\|^2 + \|w_x\|^2) \le 2(\lambda - \frac{1}{4})v(t)$$

which implies,

$$v(t) \le e^{2(\lambda-\frac{1}{4})t}v(0)$$

That is,

$$\|w\|_{H_1} \le e^{(\lambda-\frac{1}{4})t}\|w_0\|_{H_1}$$

where $w_0 = w(x,0)$ is the initial condition and $w(x,t)$ is the unique solution to (2.9).

Finally, using Young's and Agmon's inequalities in (2.18), we get,

$$\begin{aligned}\max_{x\in[0,1]} |w(x,t)|^2 &\le 2\|w\|\|w_x\| \quad (w(0)=0) \\ &\le \|w\|^2 + \|w_x\|^2 .\end{aligned}$$

Thus

$$\max_{x\in[0,1]} |w(x,t)|^2 \le e^{(\lambda-\frac{1}{4})t}\left(\|w_0\|^2 + \|w_{x,0}\|^2\right)$$

from which, we get

$$w(x,t) \to 0 \quad \text{as} \quad t\to\infty \qquad \text{for all} \quad x\in[0,1]$$

since $\lambda < \frac{1}{4}$.

2.3 Eigenfunction Expansions and Exact Solution

In this section we shall use separation of variables to find the exact solution of a simple PDE system.

Consider the diffusion equation which includes a reaction term with boundary and initial conditions,

$$
\begin{cases}
u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t),\ 0 < x < 1,\ t > 0 \\
u(0,t) = 0,\ t > 0 \\
u(1,t) = 0,\ t > 0 \\
u(x,0) = u_0(x),\ 0 < x < 1
\end{cases}
\tag{2.19}
$$

where u_0 is a continuous function over $(0,1)$. Let us find the solution to this initial boundary value problem and determine for which values of the parameter λ this system is unstable. Recall that the Lyapunove function chosen gave us $0 < \lambda < \frac{1}{4}$ for exponential stability. We assume that the solution $u(x,t)$ can be written as a product of a function of the space variable and a function of the time variable.

$$
u(x,t) = X(x)T(t). \tag{2.20}
$$

Substitution into PDE, gives,

$$
X(x)T^{'}(t) = X^{''}(x)T(t) + \lambda X(x)T(t) \tag{2.21}
$$

Division by $X(x)T(t)$ gives,

$$
\frac{T^{'}(t)}{T(t)} = \frac{X^{''}(x) + \lambda X(x)}{X(x)} \tag{2.22}
$$

In the above equation the left hand side depends only on time and the right hand side depends on the spatial variable, thus, the equality can hold only if both sides are constant, that is

$$\frac{T^{'}(t)}{T(t)} = \frac{X^{''}(x) + \lambda X(x)}{X(x)} = \sigma \quad \text{(constant)}$$

Hence,

$$T^{'}(t) = \sigma T(t), \ t > 0 \tag{2.23}$$

and

$$\begin{cases} X''(x) + (\lambda - \sigma)X(x) = 0, \ 0 < x < 1 \\ X(0) = X(1) = 0 \end{cases} \tag{2.24}$$

That is, we are dealing with a regular Sturm-Liouville problem (2.24) which has the simple eigenvalues $\lambda - \sigma_n = (n\pi)^2$, $n \geq 1$ with corresponding eigenfunctions

$$X_n(x) = A\sin(n\pi x), \ n \geq 1. \tag{2.25}$$

Now,

$$T_n(t) = e^{\sigma_n t} a_n$$

where a_n are constant,

Superposition of the product solutions gives,

$$u(x,t) = \sum_{n \geq 1} C_n e^{\sigma_n t} \sin(n\pi x)$$

that is,

$$u(x,t) = \sum_{n \geq 1} C_n e^{(\lambda - \pi^2 n^2)t} \sin(n\pi x) \tag{2.26}$$

where C_n are the Fourier coefficients of $u(x,0) = u_o(x)$ that is

$$C_n = \frac{1}{2}\int_0^1 u_0(x)\sin(n\pi x)dx \tag{2.27}$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{(\lambda-\pi^2 n^2)t}\sin(n\pi x),\ 0 < x < 1 \tag{2.28}$$

Let us look at the structure of the solution. It consists of the following elements:

· eigenvalues: $\lambda - \pi^2 n^2$

· eigenfunctions: $\sin(\pi n x)$

· effect of initial conditions: $\int_0^1 u_0(x)\sin(n\pi x)dx$

The largest eigenvalue, $\lambda - \pi^2(n=1)$, dictates the rate of growth or decay of the solution if u_0 is not orthogonal to $\sin\pi x$! If u_0 is orthogonal to $\sin\pi x$ then it is $\lambda - (2\pi)^2$, $(n=2)$ which will dictate the rate of growth or decay. Therefore in the first case we impose $\lambda < \pi^2$ for exponential stability whereas in the second case we need $\lambda < (2\pi)^2$ for exponential stability and in general if u_0 is orthogonal to $\{\sin k\pi x,$ $k = 1, 2, 3, ...N-1\}$ and not orthogonal to $\sin(N\pi x)$ then the system is exponentially stable if $\lambda < (N\pi)^2$.

2.4 Terminology

In this work we shall be using the following function spaces

Name	Description	Norm
$C^{(n)}[a,b]$	$f, f', ..f^{(n)}$ continuous functions on (a,b)	$\|f\|_\infty = \max_x \|f(x)\|$

$L^1(a,b)$ Integrable function: $\int |f(x)|\, dx < +\infty$ $\quad \|f\|_{L^1} = \int |f(x)|\, dx$

$L^2(a,b)$ Square integrable function: $\int |f(x)|^2\, dx < +\infty$ $\quad \|f\|_{L^2} = [\int |f(x)|^2\, dx]^{\frac{1}{2}}$

$H^1(a,b)$ Sobolev space: $f \in L^2$ and $f' \in L^2$ $\quad \|f\|^2_{H^1} = \|f(x)\|^2_{L^2} + \left\|f'(x)\right\|^2_{L^2}$

and in general

$H^m(a,b) = \{f \ |f, f', ..., f^{(m)} \in L^2(a,b)\}$ with the norm

$\|f\|^2_{H^m} = \sum_{j=0}^m \left\|f^{(j)}(x)\right\|^2_{L^2} = \|f\|^2_{H^{m-1}} + \left\|f^{(m)}\right\|^2_{L^2} = \left\|f'\right\|^2_{H^{m-1}} + \|f\|^2_{L^2}.$

The inner product in $L^2(a,b)$ and $H^1(a,b)$ are defined as

$$\langle f, g\rangle = \int_a^b f(x)\bar{g}(x)dx$$

and

$$\langle f, g\rangle = \int_a^b [f(x)\bar{g}(x) + f'(x)\bar{g}'(x)]dx$$

respectively.

Chapter 3

BOUNDARY CONTROL Of PARABOLIC PDEs

In this chapter we present the backstepping method to design boundary controllers stabilizing the PDE systems and show that these controllers work "irrespective" of the initial condition.

3.1 Dirichlet Condition

We consider first the case of Dirichlet actuation (where $u(1)$ is controlled), which is usually the case in many applications.

3.1.1 Backstepping: The Main Idea

Backstepping has been proved to be a remarkably elegant method for designing controllers for PDE systems. In addition, it achieves stabilization of unstable PDEs in a physically appealing way, that is, the destabilizing terms are eliminated through a change of the PDE and boundary feedback.

The main feature of backstepping is that it is capable of eliminating destabilizing effect terms that appear throughout the domain while the control is acting only from the boundary.

Let us start with the simplest unstable PDE, the reaction-diffusion equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \ 0 < x < 1, \ t > 0 \\ u(0,t) = 0, \ t > 0 \\ u(1,t) = U(t), \ t > 0 \\ u(x,0) = u_0(x), \ 0 < x < 1 \end{cases} \tag{3.1}$$

where u_0 is a continuous function, λ is a positive constant and U is the boundary control input. The open-loop system (3.1) (with $u(1,t) = 0$) is unstable with arbitrarily many unstable eigenvalues for sufficiently large λ, ($\lambda > \pi^2$). We may assume here that u_0 is not orthogonal to $\sin \pi x$ over $(0,1)$ in L_2 sense.

Since the term λu is the source of the instability, we need to eliminate this term using the backstepping method: the coordinate transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \tag{3.2}$$

maps the system (3.1) into the exponentially stable target system

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \ 0 < x < 1, \ t > 0 \\ w(0,t) = 0, \ t > 0 \\ w(1,t) = 0, \ t > 0 \\ w(x,0) = w_0(x), \ 0 < x < 1 \end{cases} \tag{3.3}$$

and results in the feedback control

$$u(1,t) = \int_0^x k(1,y)u(y,t)dy \tag{3.4}$$

Note that the transformation (3.2) is a Volterra integral transformation of the second kind, thus, invertible, so that stability of the target system translates into stability of the closed-loop system consisting of the plant plus boundary feedback.

Our goal now is to find the function $k(x,y)$ (which we call the "gain kernel"), that makes the system (3.1) with the controller behaves as the target system (3.3). It is not obvious at this point that such a transformation even exists.

3.1.2 Gain Kernel PDE

To find out what conditions $k(x, y)$ has to satisfy, we simply substitute the transformation (3.2) into the target system (3.3) and use the system equation (3.1).

To do that, we need to differentiate (3.2) with respect to x and t, which is easy once we recall the Leibnitz differentiation rule:

$$\frac{d}{dz}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} f_z(x,z)dx + f(b(z),z)b'(z) - f(a(z),z)a'(z)$$

and note that

$$\frac{d}{dx}\int_0^x f(x,y)dy = f(x,x) + \int_0^x f_x(x,y)dy$$

with appropriate conditions on the functions involved.

We also introduce the following notation:

$$\begin{aligned} k_x(x,x) &= \frac{\partial}{\partial x}k(x,y)|_{y=x} \\ k_y(x,x) &= \frac{\partial}{\partial y}k(x,y)|_{y=x} \end{aligned}$$

so that,

$$\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x).$$

Differentiating the relation (3.2) with respect to x gives,

$$w_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy$$

$$w_{xx}(x,t) = u_{xx}(x,t) - \frac{d}{dx}[k(x,x)u(x,t)] - k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x,t) - \\ &\quad k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy \end{aligned} \tag{3.5}$$

Next, we differentiate (3.2) with respect to time,

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)u_t(y,t)dy \\ w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - \int_0^x k(x,y)[u_{yy}(y,t) + \lambda u(y,t)\ ]dy \\ w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - \int_0^x k(x,y)u_{yy}(y,t)dy - \int_0^x \lambda k(x,y)u(y,t)\ dy \end{aligned}$$

Integration by parts gives,

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) + k(x,0)u_x(0,t) \\ &\quad + \int_0^x k_y(x,y)u_y(y,t)dy - \int_0^x \lambda k(x,y)u(y,t)\ dy \end{aligned}$$

and a second integration yields

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) + k(x,0)u_x(0,t) + k_y(x,x)u(x,t) - \\ &\quad k_y(x,0)u(0,t) - \int_0^x k_{yy}(x,y)u(y,t)dy - \int_0^x \lambda k(x,y)u(y,t)\ dy \end{aligned} \tag{3.6}$$

Subtracting (3.5) from (3.6) we get

$$\begin{aligned}
w_t(x,t) - w_{xx}(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) + k(x,0)u_x(0,t) + \\
&\quad k_y(x,x)u(x,t) - k_y(x,0)u(0,t) - \int_0^x k_{yy}(x,y)u(y,t)dy \\
&\quad - \int_0^x \lambda k(x,y)u(y,t)\ dy - [u_{xx}(x,t) - u(x,t)\frac{d}{dx}k(x,x) \\
&\quad -k(x,x)u_x(x,t) - k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy]
\end{aligned}$$

so,

$$\begin{aligned}
w_t(x,t) - w_{xx}(x,t) &= [\lambda + 2\frac{d}{dx}k(x,x)]u(x,t) + k(x,0)u_x(0,t) + \\
&\quad \int_0^x u(y,t)[k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)]dy
\end{aligned} \tag{3.7}$$

where we have used

$$k_x(x,x) + k_y(x,x) = \frac{d}{dx}k(x,x).$$

For the right hand side of (3.7) to be zero, we assume the following conditions

$$\begin{cases}
k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y),\ 0 < y < x < 1 \\
k(x,0) = 0,\ 0 < x < 1 \\
\lambda + 2\frac{d}{dx}k(x,x) = 0,\ 0 < x < 1
\end{cases}$$

which yield the following PDE system of hyperbolic type

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y),\ 0 < y < x < 1 \\ k(x,0) = 0,\ 0 < x < 1 \\ k(x,x) = -\frac{\lambda}{2}x,\ 0 < x < 1. \end{cases} \tag{3.8}$$

3.1.3 Converting the Gain Kernel PDE into an Integral Equation

In order to solve the above system we convert it into an equivalent integral equation using the change of variables

$$\zeta = x + y, \quad \eta = x - y, \quad k(x,y) = G(\zeta,\eta).$$

for $0 \leq \eta \leq \xi \leq 2$.

Thus,

$$\begin{aligned} k_x &= G_\zeta + G_\eta \\ k_y &= G_\zeta - G_\eta \\ k_{xx} &= G_{\zeta\zeta} + 2G_{\eta\zeta} + G_{\eta\eta} \\ k_{yy} &= G_{\zeta\zeta} - 2G_{\eta\zeta} + G_{\eta\eta} \end{aligned}$$

The kernel PDE in the new variables becomes

$$G_{\eta\zeta}(\zeta,\eta) = \frac{\lambda}{4}G(\zeta,\eta) \tag{3.9}$$

$$G(\zeta,\zeta) = 0 \tag{3.10}$$

$$G(\zeta,0) = -\frac{\lambda}{4}\zeta \tag{3.11}$$

Now, integrating (3.9) with respect to the second variable gives,

$$G_{\zeta}(\zeta,\eta) = G_{\zeta}(\zeta,0) + \int_0^{\eta} \frac{\lambda}{4}G(\zeta,s)ds = -\frac{\lambda}{4} + \int_0^{\eta} \frac{\lambda}{4}G(\zeta,s)ds. \tag{3.12}$$

Next, we integrate (3.12) with respect to the first variable to obtain,

$$G(\zeta,\eta) = G(\eta,\eta) - \frac{\lambda}{4}(\zeta-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_0^{\eta} G(\tau,s)ds\ d\tau.$$

Hence, we obtain the integral equation,

$$G(\zeta,\eta) \ = -\frac{\lambda}{4}(\zeta-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_0^{\eta} G(\tau,s)ds\ d\tau. \tag{3.13}$$

This integral equation is equivalent to PDE (3.9).

3.1.4 Method of Successive Approximations

The idea in this method is simple. Start with an initial guess for a solution of the integral equation, substitute it into the right-hand side of the equation, then use the obtained expression as the next guess in the integral equation and repeat the process. Eventually this process results in a solution of the integral equation as can be seen in details next.

Let us start with the initial guess

$$G_0(\zeta,\eta) \ = -\frac{\lambda}{4}(\zeta-\eta)$$

and define

$$G_n(\zeta,\eta) \ = \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G_{n-1}(\tau,s)ds\ d\tau,\ \ n\geq 1.$$

We shall prove by induction that

$$G_n(\zeta,\eta) \ = -(\frac{\lambda}{4})^{n+1}\frac{\zeta^n\eta^n}{n!(n+1)!}(\zeta-\eta)$$

for $n \geq 0$.

- The relation is true for $n = 0$.
- Assume the relation is true for n.

- We shall prove it is true for $n+1$,

$$
\begin{aligned}
\frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G_n(\tau,s)ds\ d\tau &= \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} -(\frac{\lambda}{4})^{n+1}\frac{\tau^n s^n}{(n)!(n+1)!}(\tau-s)ds\ d\tau \\
&= \frac{\lambda}{4}\int_{\eta}^{\xi} -(\frac{\lambda}{4})^{n+1}\frac{1}{(n)!(n+1)!}\int_{0}^{\eta}(\tau^{n+1}s^n-\tau^n s^{n+1})ds\ d\tau \\
&= -(\frac{\lambda}{4})^{n+2}\frac{1}{(n)!(n+1)!}\int_{\eta}^{\xi}[\frac{\tau^{n+1}s^{n+1}}{n+1}-\frac{\tau^n s^{n+2}}{n+2}]_{s=0}^{\eta}d\tau \\
&= -(\frac{\lambda}{4})^{n+2}\frac{1}{(n)!(n+1)!}\int_{\eta}^{\xi}[\frac{\tau^{n+1}\eta^{n+1}}{n+1}-\frac{\tau^n \eta^{n+2}}{n+2}]d\tau \\
&= -(\frac{\lambda}{4})^{n+2}\frac{1}{(n)!(n+1)!}[\frac{\tau^{n+2}\eta^{n+1}}{(n+1)(n+2)}-\frac{\tau^{n+1}\eta^{n+2}}{(n+1)(n+2)}]_{\tau=\eta}^{\xi} \\
&= -(\frac{\lambda}{4})^{n+2}\frac{1}{(n+2)!(n+1)!}[\xi^{n+2}\eta^{n+1}-\xi^{n+1}\eta^{n+2}] \\
&= -(\frac{\lambda}{4})^{n+2}\frac{\xi^{n+1}\eta^{n+1}}{(n+2)!(n+1)!}(\xi-\eta) = G_{n+1}(\xi,\eta).
\end{aligned}
$$

Hence, the relation is true for $n+1$.

- Therefore, it is true for all $n \geq 0$, and we can write the solution $G(\zeta,\eta)$ as

$$G(\zeta,\eta) = -\sum_{n=0}^{\infty}\frac{\zeta^n\eta^n(\zeta-\eta)}{n!(n+1)!}(\frac{\lambda}{4})^{n+1} \tag{3.14}$$

Now recall the first order modified *Bessel* Function

$$\mathcal{I}_1(x) = \sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n+1}}{n!(n+1)!}$$

which is solution to the *Bessel* Differential Equation

$$x^2y^{''} + xy^{'} - (x^2+1)y = 0.$$

Comparing with (3.14) we get

$$\begin{aligned}
G(\zeta,\eta) &= -(\frac{\lambda}{4})(\zeta-\eta)\sum_{n=0}^{\infty}\frac{\zeta^n\eta^n(\frac{\lambda}{4})^n}{n!(n+1)!} = -(\frac{\lambda}{4})(\zeta-\eta)\sum_{n=0}^{\infty}\frac{(\zeta\eta\frac{\lambda}{4})^n}{n!(n+1)!} \\
&= -(\frac{\lambda}{4})(\zeta-\eta)\sum_{n=0}^{\infty}\frac{(\sqrt{\zeta\eta\frac{\lambda}{4}})^{2n+1}}{n!(n+1)!\sqrt{\zeta\eta\frac{\lambda}{4}}} = -(\frac{\lambda}{4})(\zeta-\eta)\sum_{n=0}^{\infty}\frac{(\frac{1}{2}\sqrt{\zeta\eta\lambda})^{2n+1}}{n!(n+1)!\sqrt{\zeta\eta\frac{\lambda}{4}}} \\
&= -(\frac{\lambda}{4})(\zeta-\eta)\sum_{n=0}^{\infty}\frac{(\frac{1}{2}\sqrt{\zeta\eta\lambda})^{2n+1}}{n!(n+1)!\frac{\sqrt{\zeta\eta\lambda}}{2}} = -(\frac{\lambda}{4})(\zeta-\eta)\frac{\mathcal{I}_1(\sqrt{\zeta\eta\lambda})}{\frac{\sqrt{\zeta\eta\lambda}}{2}}.
\end{aligned}$$

Thus,

$$G(\zeta,\eta) = -(\frac{\lambda}{2})(\zeta-\eta)\frac{\mathcal{I}_1(\sqrt{\zeta\eta\lambda})}{\sqrt{\zeta\eta\lambda}}.$$

Returning to the original variables x , y, we obtain the kernel function,

$$k(x,y) = -\lambda y\frac{\mathcal{I}_1\sqrt{\lambda(x^2-y^2)}}{\sqrt{\lambda(x^2-y^2)}}. \tag{3.15}$$

3.1.5 Inverse Transformation

Since the Volterra integral operator of the second kind is invertible, we shall find the inverse transformation in the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy \tag{3.16}$$

where $l(x,y)$ is the transformation kernel.

To find the kernel PDE we differentiate the (3.16) with respect to x and t

$$\begin{aligned} u_t(x,t) &= w_t(x,t) + \int_0^x l(x,y) w_t(y,t) dy \\ u_t(x,t) &= w_{xx}(x,t) + \int_0^x l(x,y) w_{yy}(y,t) dy. \end{aligned}$$

Integration by parts gives,

$$u_t(x,t) = w_{xx}(x,t) + l(x,x) w_x(x,t) - l(x,0) w_x(0,t) - \int_0^x l_y(x,y) w_y(y,t) dy$$

Integrating by parts again will give,

$$\begin{aligned} u_t(x,t) &= w_{xx}(x,t) + l(x,x) w_x(x,t) - l(x,0) w_x(0,t) - \\ & \quad l_y(x,x) w(x,t) + l_y(x,0) w(0,t) + \int_0^x l_{yy}(x,y) w(y,t) dy \end{aligned}$$

so that,

$$\begin{aligned} u_t(x,t) &= w_{xx}(x,t) + l(x,x) w_x(x,t) - l(x,0) w_x(0,t) - l_y(x,x) w(x,t) \\ & \quad + \int_0^x l_{yy}(x,y) w(y,t) dy. \end{aligned} \tag{3.17}$$

Differentiating twice (3.16) with respect to x gives,

$$u_x(x,t) = w_x(x,t) + l(x,x) w(x,t) + \int_0^x I_x(x,y) w(y,t) dy$$

and

$$
\begin{aligned}
u_{xx}(x,t) &= w_{xx}(x,t) + w(x,t)\frac{d}{dx}l(x,x) + l(x,x)w_x(x,t) + \\
&\quad l_x(x,x)w(x,t) + \int_0^x l_{xx}(x,y)w(y,t)dy.
\end{aligned} \tag{3.18}
$$

Then subtracting (3.18) from (3.17) gives

$$
\begin{aligned}
u_t(x,t) - u_{xx}(x,t) &= -2w(x,t)\frac{d}{dx}l(x,x) - l(x,0)w_x(0,t) \\
&\quad + \int_0^x (l_{yy}(x,y) - l_{xx}(x,y))w(y,t)dy
\end{aligned}
$$

but,

$$
u_t(x,t) - u_{xx}(x,t) = \lambda u(x,t)
$$

so,

$$
\begin{aligned}
\lambda w(x,t) + \lambda \int_0^x l(x,y)w(y,t)dy &= -2w(x,t)\frac{d}{dx}l(x,x) - l(x,0)w_x(0,t) \\
&\quad + \int_0^x (l_{yy}(x,y) - l_{xx}(x,y))w(y,t))dy.
\end{aligned}
$$

Hence, we impose $l(x,y)$ to satisfy,

$$
\begin{cases}
l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y),\ 0 < y < x < 1 \\
l(x,0) = 0,\ 0 < x < 1 \\
l(x,x) = \frac{-\lambda}{2}x,\ 0 < x < 1
\end{cases} \tag{3.19}
$$

Comparing this PDE (3.19) with PDE (3.8) for $k(x, y)$ we see that

$$l(x, y, \lambda) = -k(x, y, -\lambda)$$

Using the properties of the *Bessel* functions $\mathcal{I}_n$,

$$\mathcal{I}_n(x) = i^{-n}\mathcal{J}_n(ix), \quad \mathcal{I}_n(ix) = i^n\mathcal{J}_n(x)$$

where $i = \sqrt{-1}$, we obtain,

$$l(x, y) = -(-\lambda)\, y \frac{\mathcal{I}_1\left(\sqrt{(-\lambda)\,(x^2 - y^2)}\right)}{\sqrt{(-\lambda)\,(x^2 - y^2)}} = -\lambda y \frac{\mathcal{I}_1\left(i\sqrt{\lambda(x^2 - y^2)}\right)}{i\sqrt{\lambda(x^2 - y^2)}}$$

that is,

$$l(x, y) = -\lambda y \frac{\mathcal{J}_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}}. \tag{3.20}$$

Hence the inverse transformation is,

$$u(x, t) = w(x, t) - \int_0^x \lambda y \frac{\mathcal{J}_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} w(y, t) dy \tag{3.21}$$

while the direct transformation is,

$$w(x, t) = u(x, t) + \int_0^x \lambda y \frac{\mathcal{I}_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} u(y, t) dy. \tag{3.22}$$

The boundary controller is therefore,

$$U(t) = -\int_0^1 k_D(y)u(y,t)dy \tag{3.23}$$

where,

$$k_D(y) = \lambda y \frac{\mathcal{I}_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \tag{3.24}$$

Since the solution to the target system (3.3) can be found explicitly and the direct and inverse transformation (3.2), (3.16) are known explicitly, it is possible to derive the explicit solution to the closed-loop system.

3.2 Neumann Actuation

In problems with thermal and chemically reacting dynamics, the natural choice is the Neumann actuation (where $u_x(1)$, or heat flux, is controlled). The Neumann controllers are obtained using the same transformation (3.2) as in the case of the Dirichlet actuation, but with the appropriate change in the boundary condition of the target system (from Dirichlet to Neumann). To illustrate the design procedure, consider the system (3.1) but with $u_x(1)$ actuated:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \ 0 < x < 1, \ t > 0 \\ u(0,t) = 0, \ t > 0 \\ u_x(1,t) = U(t), \ t > 0 \\ u(x,0) = u_0(x), \ 0 < x < 1 \end{cases} \tag{3.25}$$

where u_0 is a continuos function and U is the boundary control.

Let us use the same transformation (3.2), (3.16) as we used in the case of Dirichlet actuation. To obtain the control $u_x(1)$, we need to differentiate (3.2) with respect to x,

$$w_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy \tag{3.26}$$

and set $x = 1$. It is clear now that the target system has to have the Neumann boundary condition at $x = 1$:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t),\ 0 < x < 1,\ t > 0 \\ w(0,t) = 0,\ t > 0 \\ w_x(1,t) = 0,\ t > 0 \\ w(x,0) = w_0(x),\ 0 < x < 1 \end{cases} \tag{3.27}$$

where w_0 is a continuous function.

The controller is therefore

$$u_x(1,t) = k(1,1)u(1,t) - \int_0^1 k_x(1,y)u(y,t)dy \tag{3.28}$$

All that remains is to derive the expression for k_x from (3.15) using the property

$$\frac{d}{dx}(\mathcal{I}_n(x) = \frac{n}{x}\mathcal{I}_n(x) + \mathcal{I}_{n+1}(x)$$

of the *Bessel* functions .

Thus,

$$k_x(x,y) = -\lambda y x \frac{\mathcal{I}_2(\sqrt{\lambda(x^2-y^2)})}{x^2-y^2}. \tag{3.29}$$

Finally, the controller is

$$U(t) = -\frac{\lambda}{2}u(1,t) - \int_0^1 k_N(y)u(y,t)dy \tag{3.30}$$

where

$$k_N(y) = \lambda y \frac{\mathcal{I}_2(\sqrt{\lambda(1-y^2)})}{1-y^2} \tag{3.31}$$

3.3 Robin Condition

The Robin controllers are obtained using the same transformation (3.2) as in the case of the Dirichlet and Neumann, but with the appropriate change in the boundary condition of the target system,

Consider the system,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t),\ 0 < x < 1,\ t > 0 \\ u(0,t) + u_x(0,t) = 0,\ t > 0 \\ u(1,t) + u_x(1,t) = U(t),\ t > 0 \\ u(x,0) = u_0(x),\ 0 < x < 1 \end{cases} \tag{3.32}$$

where u_0 is a continuous function and U is the boundary control.

Using (3.2) and (3.26) we obtain,

$$w(1,t) = u(1,t) - \int_0^1 k(1,y)u(y,t)dy$$

$$w_x(1,t) = u_x(1,t) - k(1,1)u(1,t) - \int_0^1 k_x(1,y)u(y,t)dy$$

from which we get the boundary control as,

$$U(t) = -\frac{\lambda}{2}u(1,t) - \int_0^1 k_R(y)u(y,t)dy \tag{3.33}$$

where,

$$k_R(y) = \lambda y \left[\frac{\mathcal{I}_1\sqrt{\lambda(1-y^2}}{\sqrt{\lambda(1-y^2)}} + \frac{\mathcal{I}_2\sqrt{\lambda(1-y^2)}}{\lambda(1-y^2)} \right]. \tag{3.34}$$

As for the target system we get,

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \ 0 < x < 1, \ t > 0 \\ w(0,t) + w_x(0,t) = 0, \ t > 0 \\ w(1,t) + w_x(1,t) = 0, \ t > 0 \\ w(x,0) = w_0(x), \ 0 < x < 1 \end{cases} \tag{3.35}$$

where w_0 is a continuous function.

Chapter 4

PROBLEM STATEMENT

4.1 *Time independent/space dependent coefficients: the homogeneous case*

We shall consider in this section the control of the linear homogeneous reaction-advection-diffusion PDE system with a space dependent and time independent coefficients with Robin boundary conditions,

$$\begin{cases} u_t(x,t) = a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t), \ 0 < x < 1, \ t > 0 \\ \alpha u(0,t) + \beta u_x(0,t) = 0, \ t > 0 \\ \gamma u(1,t) + \delta u_x(1,t) = U(t) \ , \ t > 0 \\ u(x,0) = \phi(x), \ 0 < x < 1 \end{cases} \tag{4.1}$$

where ϕ, $c \in C[0,1]$, $a > 0$, $a \in C^2[0,1]$, $b \in C^1[0,1]$ and U is the boundary control.

When there is no control ($U(t) \equiv 0$) the system is unstable for appropriate choice of coefficient functions.

The following theorem provides us with a way to normalize the PDE system.

Theorem 3 *(Change of the independent and dependent variables)*

Let

$$\begin{cases} t = \frac{\tau}{a_0} \\ z = \sqrt{a_0}\int_0^x \frac{ds}{\sqrt{a(s)}} \\ v(z,\tau) = a^{-\frac{1}{4}}(x)u(x,t)\exp(\int_0^x \frac{b(s)}{2a(s)}ds) \end{cases} \tag{4.2}$$

where

$$a_0 = \left(\int_0^1 \frac{ds}{\sqrt{a(s)}}\right)^{-2}$$

Then v satisfies,

$$\begin{cases} v_\tau(z,\tau) = v_{zz}(z,\tau) + q(z)v(z,\tau), \ 0 < z < 1, \tau > 0 \\ \alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0, \ \tau > 0 \\ \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) = V(\tau), \ \tau > 0 \\ v(z,0) = \psi(z), \ 0 < z < 1 \end{cases} \tag{4.3}$$

where q, $\psi \in C^1(0,1)$ and

$$\begin{aligned} q(z) &= \frac{1}{a_0}\left[c(x) + \frac{a''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(a'(x))^2}{a(x)} + \frac{a'(x)b(x)}{2a(x)} - \frac{b^2(x)}{4a(x)}\right] \\ \psi(z) &= \phi(x) \\ \alpha_1 &= \{\alpha a(0) + \beta[\frac{1}{4}a'(0) - \frac{b(0)}{2}]\}a^{\frac{-3}{4}}(0) \\ \beta_1 &= \beta\sqrt{a_0}a^{\frac{-1}{4}}(0) \\ \gamma_1 &= \{\gamma a(1) + \delta[\frac{1}{4}a'(1) - \frac{b(1)}{2}]\}a^{\frac{-3}{4}}(1)\exp(-\int_0^1 \frac{b(s)}{2a(s)}ds) \\ \delta_1 &= \delta\frac{\sqrt{a_0}}{\sqrt{a(1)}}\exp(-\int_0^1 \frac{b(s)}{2a(s)}ds). \end{aligned}$$

Note that

$$V(\tau) = U(t)$$

Proof. Indeed, using the change of variables (4.2), we get after differentiating with respect to x,

$$\begin{gathered} u_x(x,t) = \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ -a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \end{gathered} \tag{4.4}$$

A second differentiation with respect to x gives,

$$
\begin{aligned}
u_{xx}(x,t) = &\tfrac{-3}{16}a^{-\frac{7}{4}}(x)\left(a'(x)\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a''(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-\tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-\tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\tfrac{b'(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \\
&a^{\frac{1}{4}}(x)a'(x)\tfrac{b(x)}{2(a(x))^2}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\left(\tfrac{b(x)}{2a(x)}\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\tfrac{\sqrt{a_0}a'(x)}{2a(x)\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&+a^{\frac{1}{4}}(x)\tfrac{a_0}{a(x)}v_{zz}(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds}
\end{aligned}
\tag{4.5}
$$

Replacing into the PDE (4.1), we obtain,

$$
\begin{aligned}
&a_0 a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_t(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
= \;& a(x)[\frac{-3}{16}a^{-\frac{7}{4}}(x)\left(a'(x)\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{-\frac{3}{4}}(x)a''(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\frac{b'(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&+a^{\frac{1}{4}}(x)a'(x)\frac{b(x)}{2\left(a(x)\right)^2}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\left(\frac{b(x)}{2a(x)}\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{\frac{-3}{4}}(x)a'(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}a'(x)}{2a(x)\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&+a^{\frac{1}{4}}(x)\frac{a_0}{a(x)}v_{zz}(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds}] + b(x)[\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&-a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\
&+c(x)a^{\frac{1}{4}}(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds}
\end{aligned}
\tag{4.6}
$$

which leads to

$$v_\tau(z,\tau) = v_{zz}(z,\tau)+\frac{1}{a_0}\left[c(x)+\frac{a''(x)}{4}-\frac{b'(x)}{2}-\frac{3}{16}\frac{(a'(x))^2}{a(x)}+\frac{a'(x)b(x)}{2a(x)}-\frac{b^2(x)}{4a(x)}\right]v(z,\tau)$$

that is,

$$v_t(z,\tau) = v_{zz}(z,\tau) + q(z)v(z,\tau) \tag{4.7}$$

where

$$q(z) = \frac{1}{a_0}\left[c(x)+\frac{a''(x)}{4}-\frac{b'(x)}{2}-\frac{3}{16}\frac{(a'(x))^2}{a(x)}+\frac{a'(x)b(x)}{2a(x)}-\frac{b^2(x)}{4a(x)}\right].$$

As for the boundary condition,

$$\alpha u(0,t) + \beta u_x(0,t) = 0$$

it leads to,

$$\alpha a^{\frac{1}{4}}(0)v(0,\tau) + \beta[\frac{a^{\frac{-3}{4}}(0)}{4}a'(0)v(0,\tau) - \frac{a^{\frac{1}{4}}(0)b(0)}{2a(0)}v(0,\tau) + \frac{a^{\frac{1}{4}}(0)\sqrt{a_0}}{\sqrt{a(0)}}v_z(0,\tau)] = 0$$

$$\{\alpha a(0) + \beta[\frac{1}{4}a'(0) - \frac{b(0)}{2}]\}a^{\frac{-3}{4}}(0)v(0,\tau) + [\beta a_0^{\frac{1}{2}}a^{\frac{-1}{4}}(0)]v_z(0,\tau) = 0 \tag{4.8}$$

from which we get,

$$\alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0 \tag{4.9}$$

where

$$\begin{cases} \alpha_1 = \{\alpha a(0) + \beta[\frac{1}{4}a'(0) - \frac{b(0)}{2}]\}a^{\frac{-3}{4}}(0) \\ \beta_1 = \beta a_0^{\frac{1}{2}} a^{\frac{-1}{4}}(0). \end{cases} \tag{4.10}$$

Similarly,

$$U(t) = \gamma u(1,t) + \delta u_x(1,t)$$

leads to,

$$\begin{aligned} U(t) &= \gamma a^{\frac{1}{4}}(1)e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v(1,\tau) + \delta[\frac{1}{4}a^{-\frac{3}{4}}(1)a'(1)e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v(1,\tau) \\ &\quad -a^{\frac{1}{4}}(1)\frac{b(1)}{2a(1)}e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v(1,\tau) + a^{\frac{1}{4}}(1)\frac{\sqrt{a_0}}{\sqrt{a(1)}}e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v_z(1,\tau) \\ V(\tau) &= \{\gamma a(1) + \delta[\frac{1}{4}a'(1) - \frac{1}{2}b(1)]\}a^{-\frac{3}{4}}(1)e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v(1,\tau) \\ &\quad +\delta\frac{\sqrt{a_0}}{\sqrt{a(1)}}e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}v_z(1,\tau) \end{aligned} \tag{4.11}$$

that is,

$$V(\tau) = \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) \tag{4.12}$$

where,

$$\begin{cases} \gamma_1 = \{\gamma a(1) + \delta[\frac{1}{4}a'(1) - \frac{b(1)}{2}]\}a^{\frac{-3}{4}}(1)e^{-\int_0^1 \frac{b(s)}{2a(s)}ds} \\ \delta_1 = \delta\frac{\sqrt{a_0}}{\sqrt{a(1)}}e^{-\int_0^1 \frac{b(s)}{2a(s)}ds}. \end{cases} \tag{4.13}$$

Hence, (4.3) which completes the proof. ■

Theorem 4 *Let* $L_0 = \frac{\partial^2}{\partial z^2} + q$ *and* $L_1 = \frac{\partial^2}{\partial z^2} + \lambda$,*where* $q \in C^1[0,1]$ *and* λ *is an arbitrary constant.*

Let $\mathbb{K} : H^2(0,1) \to H^2(0,1)$ *be the operator defined by,*

$$(\mathbb{K}v)(z,t) = \int_0^z k(z,y)v(y,t)dy \tag{4.14}$$

where k *solves the Goursat problem,*

$$\begin{cases} k_{zz}(z,y) - k_{yy}(z,y) = [q(y)-\lambda]k(z,y),\ 0 \le y \le z \le 1 \\ \frac{d}{dz}k(z,z) = -\frac{1}{2}[q(z)-\lambda],\ 0 \le z \le 1 \\ \alpha_1 k(z,0) + \beta_1 k_y(z,0) = 0,\ 0 \le z \le 1 \\ k(0,0) = 0 \end{cases} \tag{4.15}$$

Then the operator $1-\mathbb{K}$ *is :*

(i) a transmutation operator for the pair of operators $\{L_0, L_1\}$ *i.e.,* $(1-\mathbb{K})L_0 = L_1(1-\mathbb{K})$

(ii) bounded, invertible, with a bounded inverse $(1-\mathbb{K})^{-1} = 1+\mathbb{L}$, $\mathbb{L} : H^2(0,1) \to H^2(0,1)$, *defined by*

$$(\mathbb{L}\hat{v})(z,\tau) = \int_0^z l(z,y)\hat{v}(y,\tau)dy \tag{4.16}$$

where l *solves the Goursat problem,*

$$\begin{cases} l_{zz}(z,y) - l_{yy}(z,y) = -[q(y)-\lambda]l(z,y),\ 0 \le y \le z \le 1 \\ \frac{d}{dz}l(z,z) = -\frac{1}{2}[q(z)-\lambda],\ 0 \le z \le 1 \\ \alpha_1 l(z,0) + \beta_1 l_y(z,0) = 0,\ 0 \le z \le 1 \\ l(0,0) = 0 \end{cases} \tag{4.17}$$

Proof. (i) We shall prove that $1-\mathbb{K}$ is a transmutation operator.

We have,

$$\begin{aligned}
(1-\mathbb{K})L_0 v &= (1-\mathbb{K})(\frac{\partial^2}{\partial z^2}+q)v \\
&= (\frac{\partial^2}{\partial z^2}+q)v-\int_0^z k(z,y)(v_{zz}(y,\tau)+q(y)v(y,\tau))dy \\
&= v_{zz}(z,\tau)+q(z)v(z,\tau)-\int_0^z k(z,y)(v_{yy}(y,\tau)+qv(y,\tau))dy \\
&= v_{zz}(z,\tau)+q(z)v(z,\tau)-\int_0^z k(z,y)v_{yy}(y,\tau)dy+\int_0^z q(y)k(z,y)v(y,\tau)dy \\
&= v_{zz}(z,\tau)+q(z)v(z,\tau)-k(z,z)v_z(z,\tau)+k(z,0)v_z(0,\tau)+k_y(z,z)v(z,\tau) \\
&\quad -k_y(z,0)v(0,\tau)-\int_0^z k_{yy}(z,y)v(y,\tau)dy-\int_0^z q(y)k(z,y)v(y,\tau)dy.
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
L_1(1-\mathbb{K})v &= (\frac{\partial^2}{\partial z^2}+\lambda)(1-\mathbb{K})v \\
&= (\frac{\partial^2}{\partial z^2}+\lambda)(v(z,\tau)-\int_0^z k(z,y)v(y,\tau)dy) \\
&= v_{zz}(z,\tau)+\lambda v(z,\tau)-\lambda\int_0^z k(z,y)v(z,\tau)-\frac{\partial^2}{\partial z^2}(\int_0^z k(z,y)v(y,\tau)dy) \\
&= v_{zz}(z,\tau)+\lambda v(z,\tau)-\lambda\int_0^z k(z,y)v(z,\tau)dy-v(z,z)\frac{d}{dz}k(z,z) \\
&\quad -k(z,z)v_z(z,\tau)-k_z(z,z)v(z,\tau)-\int_0^z k_{zz}(z,y)v(y,\tau)dy
\end{aligned} \tag{4.19}$$

so that,

$$
\begin{aligned}
& (1-\mathbb{K})L_0 v - L_1(1-\mathbb{K})v \\
= \;& v_{zz}(z,\tau) + q(z)v(z,\tau) - k(z,z)v_z(z,\tau) + k(z,0)v_z(0,\tau) + k_y(z,z)v(z,\tau) - \\
& k_y(z,0)v(0,\tau) - \int_0^z k_{yy}(z,y)v(y,\tau)dy - \int_0^z q(y)k(z,y)v(y,\tau)dy - [v_{zz}(z,\tau) + \\
& \lambda v(z,\tau) - \lambda \int_0^z k(z,y)v(z,\tau) - v(z,z)\frac{d}{dz}k(z,z) - k(z,z)v_z(z,\tau) \\
& -k_z(z,z)v(z,\tau) - \int_0^z k_{zz}(z,y)v(y,\tau)dy] \\
= \;& \int_0^z \left[k_{zz}(z,y) - k_{yy}(z,y) - (q(y)-\lambda)k(z,y)\right] v(z,\tau)dy + \\
& [2\frac{d}{dz}k(z,z) + q(z) - \lambda]v(z,\tau) + [k(z,0)v_z(0,\tau) - k_y(z,0)v(0] \\
= \;& 0
\end{aligned}
$$

for all $v \in H^2$ if we choose k to satisfy (4.15). Thus

$$(1-\mathbb{K})L_0 = L_1(1-\mathbb{K})$$

Similarly, we have

$$\begin{aligned}
(1+\mathbb{L})L_1\hat{v} &= (1+\mathbb{L})(\frac{\partial^2}{\partial z^2}+\lambda)\hat{v} \\
&= (\frac{\partial^2}{\partial z^2}+\lambda)\hat{v}+\int_0^z l(z,y)(\hat{v}_{zz}(y,\tau)+\lambda\hat{v}(y,\tau))dy \\
&= \hat{v}_{zz}(z,\tau)+\lambda\hat{v}(z,\tau)+\int_0^z l(z,y)(\hat{v}_{zz}(y,\tau)+\lambda\hat{v}(y,\tau))dy \\
&= \hat{v}_{zz}(z,\tau)+\lambda\hat{v}(z,\tau)+\int_0^z l(z,y)\hat{v}_{zz}(y,\tau)dy+\int_0^z \lambda l(z,y)\hat{v}(y,\tau)dy \\
&= \hat{v}_{zz}(z,\tau)+\lambda\hat{v}(z,\tau)+l(z,z)\hat{v}_z(z,\tau)-l(z,0)\hat{v}_z(0,\tau)-l_y(z,z)\hat{v}(z,\tau) \\
&\quad +l_y(z,0)\hat{v}(0,\tau)+\int_0^z l_{yy}(z,y)\hat{v}(y,\tau)dy+\int_0^z \lambda l(z,y)\hat{v}(y,\tau)dy
\end{aligned}$$

and

$$\begin{aligned}
L_0(1+\mathbb{L})\hat{v} &= (\frac{\partial^2}{\partial z^2}+q)(1+\mathbb{L})\hat{v} \\
&= (\frac{\partial^2}{\partial z^2}+q)(\hat{v}(z,\tau)+\int_0^z l(z,y)\hat{v}(y,\tau)dy) \\
&= \hat{v}_{zz}(z,\tau)+q(z)\hat{v}(z,\tau)+\int_0^z l(z,y)q(y)\hat{v}(z,\tau)+\frac{\partial^2}{\partial z^2}(\int_0^z l(z,y)\hat{v}(y,\tau)dy) \\
&= \hat{v}_{zz}(z,\tau)+q(z)\hat{v}(z,\tau)+\int_0^z l(z,y)q(y)\hat{v}(z,\tau)+\hat{v}(z,z)\frac{d}{dz}l(z,z)+ \\
&\quad l(z,z)\hat{v}_z(z,\tau)+l_z(z,z)\hat{v}(z,\tau)+\int_0^z l_{zz}(z,y)\hat{v}(y,\tau)dy
\end{aligned}$$

so that

$$
\begin{aligned}
& (1+\mathbb{L})L_1\hat{v} - L_0(1+\mathbb{L})\hat{v} \\
= \; & \hat{v}_{zz}(z,\tau) + \lambda\hat{v}(z,\tau) + l(z,z)\hat{v}_z(z,\tau) - l(z,0)\hat{v}_z(0,\tau) - l_y(z,z)\hat{v}(z,\tau) \\
& + l_y(z,0)\hat{v}(0,\tau) + \int_0^z l_{yy}(z,y)\hat{v}(y,\tau)dy + \int_0^z \lambda l(z,y)\hat{v}(y,\tau)dy - \\
& [\hat{v}_{zz}(z,\tau) + q(z)\hat{v}(z,\tau) + \int_0^z l(z,y)q(y)\hat{v}(z,\tau) + \hat{v}(z,z)\frac{d}{dz}l(z,z) + \\
& l(z,z)\hat{v}_z(z,\tau) + l_z(z,z)\hat{v}(z,\tau) + \int_0^z l_{zz}(z,y)\hat{v}(y,\tau)dy] \\
= \; & \int_0^z [l_{yy}(z,y) - l_{zz}(z,y) - (q(y)-\lambda)l(z,y)]\hat{v}(z,\tau)dy + \\
& [-2\frac{d}{dz}l(z,z) - (q(z)-\lambda)\hat{v}(z,\tau)] + [l_y(z,0)\hat{v}(0,\tau) - l(z,0)\hat{v}_z(0,\tau)] \\
= \; & 0
\end{aligned}
$$

for all $\hat{v} \in H^2$ If we choose l to satisfy (4.17).

Thus,

$$(1+\mathbb{L})L_1 = L_0(1+\mathbb{L}). \tag{4.20}$$

(ii) $1-\mathbb{K}$ is bounded with bounded inverse, since it is a Volterra operator of the second kind.

Let $1-\mathbb{K}$ be the transmutation defined by

$$\hat{v}(z,\tau) = v(z,t) - \int_0^z k(z,y)v(y,\tau)dy$$

and its inverse $1 + \mathbb{L}$ be

$$v(z,\tau) = \hat{v}(z,\tau) + \int_0^z k(z,y)\hat{v}(y,\tau)dy$$

We shall transform the system (4.3), by applying $1 - \mathbb{K}$ to both sides of the PDE in (4.3), then, interchanging with $\partial/\partial\tau$ and using the transmutation rule we get,

$$(1-\mathbb{K})\tfrac{\partial}{\partial\tau}v(z,\tau) = (1-\mathbb{K})(\tfrac{\partial^2}{\partial z^2}v(z,\tau) + q(z)v(z,\tau)),\ 0 < z < 1,\ \tau > 0$$
$$\tfrac{\partial}{\partial\tau}(1-\mathbb{K})v = (1-\mathbb{K})(\tfrac{\partial^2}{\partial z^2} + q(z))v$$
$$\tfrac{\partial}{\partial\tau}(1-\mathbb{K})v = (1-\mathbb{K})\mathbb{L}_0 v$$
$$\tfrac{\partial}{\partial\tau}(1-\mathbb{K})v = \mathbb{L}_1(1-\mathbb{K})v$$
$$\tfrac{\partial}{\partial\tau}\hat{v} = \mathbb{L}_1\hat{v}$$
$$\tfrac{\partial}{\partial\tau}\hat{v} = (\tfrac{\partial^2}{\partial z^2} + \lambda)\hat{v}.$$

As for the boundary conditions, we proceed as follows.

Since,

$$v = (1+\mathbb{L})\hat{v},\ \hat{v} = (1-\mathbb{K})v$$

we have

$$\begin{aligned} v(z,\tau) &= \hat{v}(z,\tau) + \int_0^z l(z,y)\hat{v}(y,\tau)dy \\ v_z(z,\tau) &= \hat{v}_z(z,\tau) + l(z,z)\hat{v}(z,\tau) + \int_0^z l_z(z,y)\hat{v}(y,\tau)dy \end{aligned}$$

$$
\begin{aligned}
v(0,\tau) &= \hat{v}(0,\tau) \\
v_z(0,\tau) &= \hat{v}(0,\tau) + l(0,0)\hat{v}(0,\tau)
\end{aligned}
$$

so that,

$$\alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = \alpha_1 \hat{v}(0,\tau) + \beta_1(\hat{v}(0,\tau)$$

thus,

$$\alpha_1 \hat{v}(0,\tau) + \beta_1 \hat{v}(0,\tau) = 0.$$

We shall take the other boundary condition to be

$$\gamma_1 \hat{v}(1,\tau) + \delta_1 \hat{v}_z(1,\tau) = 0$$

leading to the target system,

$$\begin{cases} \hat{v}_\tau(z,\tau) = \hat{v}_{zz}(z,\tau) + \lambda \hat{v}(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 \hat{v}(0,\tau) + \beta_1 \hat{v}_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 \hat{v}(1,\tau) + \delta_1 \hat{v}_z(1,\tau) = 0,\ \tau > 0 \\ \hat{v}(z,0) = \hat{\psi}(z),\ 0 < z < 1 \end{cases} \tag{4.21}$$

where $\hat{\psi} = (1 - \mathbb{K})\psi \in C^1(0,1)$.

But $v = (1 + \mathbb{L})\hat{v}$ that is,

$$v(z,\tau) = \hat{v}(z,\tau) + \int_0^z l(z,y)\hat{v}(y,\tau)dy$$

so,

$$\begin{aligned} v(z,t) &= \hat{v}(1,t) + \int_0^z l(1,y)\hat{v}(y,\tau)dy \\ v_z(1,\tau) &= \hat{v}_z(1,\tau) - l(1,1)\hat{v}(1,\tau) + \int_0^z l_z(1,y)\hat{v}(y,\tau)dy \end{aligned}$$

from which we get the boundary control

$$\begin{aligned} V(\tau) &= \gamma_1[\hat{v}(1,\tau) + \int_0^1 l(1,y)\hat{v}(y,\tau)dy] + \delta_1[\hat{v}_z(1,\tau) + \\ & \quad l(1,1)\hat{v}(1,\tau) + \int_0^1 l_z(1,y)\hat{v}(y,\tau)dy] \end{aligned}$$

$$V(\tau) = \delta_1 l(1,1)\hat{v}(1,\tau) + \int_0^1 [\gamma_1 l(1,y) + \delta_1 l_z(1,y)]\hat{v}(y,\tau)dy$$

■

The kernel PDE (4.15) cannot be solved in closed form. So we shall search a solution as a series. For this, we first convert (4.15) into an integral equation. Introducing the change of variables

$$\xi = z + y \quad , \eta = z - y \quad k(z,y) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = G(\xi,\eta) \tag{4.22}$$

we have

$$\begin{aligned} k_z &= G_\xi + G_\eta,\ k_{zz} = G_{\xi\xi} + 2G_{\eta\xi} + G_{\eta\eta} \\ k_y &= G_\xi - G_\eta\ , k_{yy} = G_{\xi\xi} - 2G_{\eta\xi} + G_{\eta\eta}. \end{aligned}$$

The PDE system in the new variables is

$$\begin{cases} 4G_{\eta\xi}(\xi,\eta) = [q(\frac{\xi-\eta}{2}) - \lambda]G(\xi,\eta),\ 0 \le \eta \le \xi \le 2 \\ \alpha_1 G(\xi,\xi) + \beta_1\left[G_\xi(\xi,\xi) - G_\eta(\xi,\xi)\right] = 0,\ 0 \le \xi \le 2, \alpha_1^2+\beta_1^2 \neq 0 \\ \frac{\partial}{\partial\xi}G\left(\xi,0\right) = -\frac{1}{4}[q(\frac{\xi}{2}) - \lambda],\ 0 \le \xi \le 2 \end{cases} \tag{4.23}$$

$$\begin{cases} G_{\eta\xi}(\xi,\eta) = \frac{1}{4}[q(\frac{\xi-\eta}{2}) - \lambda]G(\xi,\eta),\ 0 \le \eta \le \xi \le 2 \\ G(\xi,\xi) = -\beta_1,\ 0 \le \xi \le 2 \\ \left[G_\xi(\xi,\xi) - G_\eta(\xi,\xi)\right] = \alpha_1,\ 0 \le \xi \le 2,\ \alpha_1^2+\beta_1^2 \neq 0 \\ G_\xi\left(\xi,0\right) = -\frac{1}{4}[q(\frac{\xi}{2}) - \lambda],\ 0 \le \xi \le 2 \end{cases} \tag{4.24}$$

note that at $y=0$ we have $\eta = z = \xi$ and at $y=z$ we have $\eta = 0$ and $\xi = 2z$ imply $z = \frac{\xi}{2}$.

Now integrating (4.24) with respect to η gives,

$$\begin{aligned} G_\xi(\xi,\eta) &= G_\xi(\xi,0) + \frac{1}{4}\int_0^\eta [q(\frac{\xi-s}{2}) - \lambda]G(\xi,s)ds \\ &= \frac{1}{4}[\lambda - q(\frac{\xi}{2})] + \frac{1}{4}\int_0^\eta [q(\frac{\xi-s}{2}) - \lambda]G(\xi,s)ds \end{aligned} \tag{4.25}$$

Next we integrate (4.25) with respect to ξ to get

$$G(\xi,\eta) = G(\eta,\eta) + \frac{1}{4}\int_\eta^\xi [\lambda - q(\frac{s}{2})]ds + \frac{1}{4}\int_\eta^\xi \int_0^\eta [q(\frac{\tau-s}{2}) - \lambda]G(\tau,s)ds\ d\tau$$

that is,

$$G(\xi,\eta) \ = -\beta_1 + \frac{1}{4}\int_\eta^\xi [\lambda - q(\frac{s}{2})]ds + \frac{1}{4}\int_\eta^\xi \int_0^\eta [q(\frac{\tau-s}{2}) - \lambda]G(\tau,s)ds\ d\tau \tag{4.26}$$

By the method of successive approximation we can show that this equation has a unique continuous solution.

Indeed we have,

Lemma 5 *Let $q \in C^1[0,1]$. Problem (4.15) has a unique solution $k(z,y)$ which is twice continuously differentiable in $0 \leq y \leq z \leq 1$ given by*

$$k(z,y) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = G(\xi,\eta) = \sum_{n\geq 0} G_n(\xi,\eta)$$

where

$$G_0(\xi,\eta) = -\beta_1 + \frac{1}{4}\int_\eta^\xi \left(\lambda - q(\frac{s}{2})\right) ds$$

and

$$G_{n+1}(\xi,\eta) \ = \frac{1}{4}\int_\eta^\xi \int_0^\eta \left(q(\frac{\tau-s}{2}) - \lambda\right) G_n(\tau,s)ds\ d\tau,\ n \geq 1.$$

The series

$$\sum_{n=0}^{\infty} G_n(\xi,\eta)$$

converges absolutely and uniformly in $0 \leq \eta \leq \xi \leq 2$ and its sum $G(\xi,\eta)$ is the continuous solution to (4.26). G is twice continuously differentiable because $q \in C^1[0,1]$.

Proof. Let us start with initial guess,

$$G_0(\xi,\eta) = -\beta_1 + \frac{1}{4}\int_\eta^\xi [\lambda - q(\frac{s}{2})]ds \tag{4.27}$$

and define

$$G_n(\xi,\eta) \ = \frac{1}{4}\int_\eta^\xi \int_0^\eta [q(\frac{\tau - s}{2}) - \lambda] G_{n-1}(\tau, s) ds\ d\tau. \tag{4.28}$$

We shall prove by induction that,

$$|G_n(\xi,\eta)| \leq \left(\frac{M_1}{4}\right)^n \frac{M}{(n!)^2}\eta^n \xi^n \tag{4.29}$$

for all $n \geq 0$ where,

$$M_1 = \sup_{x\in[0,1]} |\lambda - q(x)|$$

and

$$M = |\beta_1| + \frac{M_1}{2}$$

- (4.29) is true for $n = 0$ since,

$$\begin{aligned} |G_0(\xi,\eta)| &\leq |\beta_1| + \frac{M_1}{4}(\xi - \eta) \\ &= |\beta_1| + \frac{M_1}{2} = M \end{aligned}$$

- Assume it is true for n, we shall prove it true for $n+1$.

We have

$$\begin{aligned}
G_{n+1}(\xi,\eta) &= \frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta}[q(\frac{\tau-s}{2})-\lambda]G_n(\tau,s)ds\ d\tau \\
|G_{n+1}(\xi,\eta)| &\leq \frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} M_1\left(\frac{M_1}{4}\right)^n \frac{M}{(n!)^2}\tau^n s^n ds d\tau \\
&= \left(\frac{M_1}{4}\right)^{n+1}\frac{M}{(n!)^2}\int_{\eta}^{\xi}\frac{\eta^{n+1}}{n+1}\tau^n d\tau \\
&= \left(\frac{M_1}{4}\right)^{n+1}\frac{M}{(n!)^2}\frac{\eta^{n+1}}{n+1}\frac{(\xi-\eta)^{n+1}}{n+1} \\
&\leq \left(\frac{M_1}{4}\right)^{n+1}\frac{M}{[(n+1)!]^2}\xi^{n+1}\eta^{n+1}.
\end{aligned}$$

that is (4.29) is true for $n+1$.

- Hence the inequality (4.29) is true for all $n \geq 0$.

- These estimates show that the series

$$\sum_{n=0}^{\infty} G_n(\xi,\eta)$$

converges absolutely and uniformly in $0 \leq \eta \leq \xi \leq 2$, and its sum $G(\xi,\eta)$ is the continuous solution of (4.26) . Morever, it follows from that G is twice continuously differentiable because $q \in C^1[0,1]$.

■

Theorem 6 *The system*

$$\begin{cases} \hat{v}_\tau(z,\tau) = \hat{v}_{zz}(z,\tau) + \lambda \hat{v}(z,\tau), \ 0 < z < 1, \ \tau > 0 \\ \alpha_1 \hat{v}(0,\tau) + \beta_1 \hat{v}_z(0,\tau) = 0, \ \tau > 0 \\ \gamma_1 \hat{v}(1,\tau) + \delta_1 \hat{v}_z(1,\tau) = 0, \ \tau > 0 \\ \hat{v}(z,0) = \hat{\psi}(z), \ 0 < z < 1 \end{cases}$$

where $\hat{\psi}$ is a continuous function is exponentially stable if $\lambda < \omega_{n_0}^2$ where n_0 is the first index for which $a_{n_0} \neq 0$, that is $a_1 = a_2 ...... a_{n_0-1} = 0$, and $a_{n_0} \neq 0$,where,

$$a_n = \frac{\left\langle \hat{\psi}, Z_n \right\rangle}{\|Z_n\|^2}$$

$\langle .,. \rangle$ is the $L^2[0,1]$ inner product and $\|f\|^2 = \langle f, f \rangle$. Z_n eigenfunctions corresponding to the eigenvalues ω_n^2, $n \geq 1$, $\alpha_1^2 + \beta_1^2 = 1$, $\gamma_1^2 + \delta_1^2 = 1$. and

$$|\hat{v}(z,\tau)| \leq A e^{-(\omega_{n_0}^2 - \lambda)\tau}, \quad A \text{ constant}, \ z \in (0,1)$$

Proof. Let

$$\hat{v}(z,\tau) = Z(z)T(\tau)$$

by using separation of variables, we have

$$T^{'}(t)Z(z) = Z^{''}(z)T(t) + \lambda Z(z)T(t)$$

thus,

$$\frac{T^{'}(t)}{T(t)} = \frac{Z^{''}(z)}{Z(z)} + \lambda = -\sigma$$

(σ separation constant). Thus,

$$\begin{aligned} T^{'} &= -\sigma T \\ Z^{''} + \omega^2 Z &= 0 \end{aligned}$$

where $\omega^2 = \lambda + \sigma$.

From the boundary condition we get,

$$\begin{cases} \alpha_1 Z(0)T + \beta_1 Z^{'}(0)T = 0 \\ \gamma_1 Z(1)T + \delta_1 Z^{'}(1)T = 0 \end{cases}$$

Hence, we are dealing with the regular Sturm-Liouville problem,

$$\begin{cases} Z^{''} + \omega^2 Z = 0, \;\; z \in (0,1) \\ \alpha_1 Z(0) + \beta_1 Z^{'}(0) = 0 \\ \gamma_1 Z(1) + \delta_1 Z^{'}(1) = 0 \end{cases} \tag{4.30}$$

which is known to have an infinite sequence of simple eigenvalues ω_n^2 satisfying

$$\omega_1^2 < \omega_2^2 < \omega_3^2 < ... < \omega_n^2 < .... \quad \uparrow \infty \; as \; n \to \infty$$

Corresponding to each eigenvalue ω_n^2, there is a unique independent eigenfunction Z_n. These eigenfunctions are orthogonal with respect to the $L^2[0,1]$ inner product. For completeness we shall find the eigenfunctions as follows:

The general solution of the differential equation in (4.30) is

$$Z(z) = a \cos \omega z + b \sin \omega z$$

which gives

$$Z^{'}(z) = -a\omega \sin \omega z + b\omega \cos \omega z$$

so that the boundary conditions yield,

$$\begin{cases} \alpha_1 a + \beta_1 b\omega = 0 \\ \gamma_1(a \cos \omega + b \sin \omega) + \delta_1(-a\omega \sin \omega + b\omega \cos \omega) = 0 \end{cases}$$

$$\begin{cases} \alpha_1 a + \beta_1 b\omega = 0 \\ (\gamma_1 \cos \omega - \delta_1 \omega \sin \omega)a + (\delta_1 \sin \omega + \delta_1 \omega \cos \omega)b = 0 \end{cases}$$

Thus, to have a nontrivial solution a necessary and sufficient condition is

$$\begin{vmatrix} \alpha_1 & \beta_1 \omega \\ \gamma_1 \cos \omega - \delta_1 \omega \sin \omega & \delta_1 \sin \omega + \delta_1 \omega \cos \omega \end{vmatrix} = 0$$

This constitutes the characteristic equation from which we get the eigenvalues ω_n^2, $n \geq 1$.

Thus,

$$\sigma_n = \omega_n^2 - \lambda$$

and

$$T^{'} = -\sigma_n T$$

leads to

$$T = T_n = e^{-\sigma_n \tau} c_n = c_n e^{-(\omega_n^2 - \lambda)\tau}.$$

Therefore,

$$\hat{v}_n(z,\tau) = Z_n(z) T_n(\tau) = c_n e^{-(\omega_n^2 - \lambda)\tau} Z_n(z)$$

and the superposition principle gives,

$$\hat{v}(z,\tau) = \sum_{n \geq 1} a_n e^{-(\omega_n^2 - \lambda)\tau} Z_n(z)$$

but

$$\hat{v}(z,0) = \hat{\psi}(z) = \sum_{n \geq 1} a_n Z_n(z)$$

where

$$a_n = \frac{\left\langle \hat{\psi}, Z_n \right\rangle}{\left\| Z_n \right\|^2}, \ n \geq 1$$

so $\hat{v}(z,\tau)$ is completely determined.

For exponential stability, it is enough to have $\omega_{n_0}^2 - \lambda > 0$ where n_0 is the first index for which $a_{n_0} \neq 0$. In that case $\hat{v}(z,\tau) \longrightarrow 0$ as $\tau \longrightarrow \infty$ uniformly in $z \in [0,1]$ since,

$$\hat{v}(z,\tau) = \sum_{n \geq n_0} a_n e^{-(\omega_n^2 - \lambda)\tau} Z_n(\tau)$$

we have

$$|\hat{v}(z,\tau)| \leq A e^{-(\omega_{n_0}^2 - \lambda)\tau}, \quad A \text{ is a positive constant.}$$

■

4.2 Time independent/space dependent coefficients: the nonhomogeneous case

Consider the non homogeneous one-dimensional linear reaction-advection-diffusion PDE system

$$\begin{cases} u_t(x,t) = a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t) + F(x,t), \ 0 < x < 1, \ t > 0 \\ \alpha u(0,t) + \beta u_x(0,t) = 0, \ t > 0 \\ \gamma u(1,t) + \delta u_x(1,t) = U(t), \ t > 0 \\ u(x,0) = \phi(x), \ 0 < x < 1 \end{cases} \tag{4.31}$$

where $F(x,t) \to 0$ exponentially in t as $t \to \infty$, uniformly in $x \in (0,1)$, ϕ, $c \in C[0,1]$, $a > 0$, $a \in C^2[0,1]$, $b \in C^1[0,1]$ and U is the boundary control.

Theorem 7 *The change of independent variables and dependent variables given in*

(4.2) leads v to satisfy,

$$\begin{cases} v_\tau(z,\tau) = v_{zz}(z,\tau) + q(z)v(z,\tau) + H(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) = V(\tau), control \\ v(z,0) = \psi(z) \end{cases} \tag{4.32}$$

where $H(z,\tau) \to 0$ *exponentially as* $\tau \to \infty$*, uniformly in* $z \in (0,1)$*. Where* $q,\ \psi \in C^1(0,1)$ *and*

$$\begin{aligned} q(z) &= \frac{1}{a_0}\left[c(x) + \frac{a''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(a'(x))^2}{a(x)} + \frac{a'(x)b(x)}{2a(x)} - \frac{b^2(x)}{4a(x)}\right] \\ \psi(z) &= \phi(x) \\ \alpha_1 &= \{\alpha a(0) + \beta[\frac{1}{4}a'(0) - \frac{b(0)}{2}]\}a^{\frac{-3}{4}}(0) \\ \beta_1 &= \beta\sqrt{a_0}a^{\frac{-1}{4}}(0) \\ \gamma_1 &= \{\gamma a(1) + \delta[\frac{1}{4}a'(1) - \frac{b(1)}{2}]\}a^{\frac{-3}{4}}(1)\exp(-\int_0^1 \frac{b(s)}{2a(s)}ds) \\ \delta_1 &= \delta\frac{\sqrt{a_0}}{\sqrt{a(1)}}\exp(-\int_0^1 \frac{b(s)}{2a(s)}ds). \end{aligned}$$

Note that,

$$V(\tau) = U(t)$$

Proof. Differentiating equation (4.2) with respect to x gives

$$\begin{gathered} u_x(x,t) = \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ -a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \end{gathered} \tag{4.33}$$

A second differentiation with respect to x gives,

$$\begin{aligned} u_{xx}(x,t) = {} & \tfrac{-3}{16}a^{-\frac{7}{4}}(x)\left(a'(x)\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a''(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -\tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -\tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\tfrac{b'(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \\ & a^{\frac{1}{4}}(x)a'(x)\tfrac{b(x)}{2(a(x))^2}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\left(\tfrac{b(x)}{2a(x)}\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \tfrac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -a^{\frac{1}{4}}(x)\tfrac{b(x)}{2a(x)}\tfrac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\tfrac{\sqrt{a_0}a'(x)}{2a(x)\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & +a^{\frac{1}{4}}(x)\tfrac{a_0}{a(x)}v_{zz}(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \end{aligned} \tag{4.34}$$

Replacing into the PDE (4.31), we obtain,

$$\begin{aligned} & \alpha a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_t(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ = {} & a(x)[\frac{-3}{16}a^{-\frac{7}{4}}(x)\left(a'(x)\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{-\frac{3}{4}}(x)a''(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\frac{b'(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & +a^{\frac{1}{4}}(x)a'(x)\frac{b(x)}{2\left(a(x)\right)^2}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\left(\frac{b(x)}{2a(x)}\right)^2 v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + \frac{1}{4}a^{\frac{-3}{4}}(x)a'(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} - a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}a'(x)}{2a(x)\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & +a^{\frac{1}{4}}(x)\frac{a_0}{a(x)}v_{zz}(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds}] + b(x)[\frac{1}{4}a^{-\frac{3}{4}}(x)a'(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & -a^{\frac{1}{4}}(x)\frac{b(x)}{2a(x)}v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + a^{\frac{1}{4}}(x)\frac{\sqrt{a_0}}{\sqrt{a(x)}}v_z(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} \\ & +c(x)a^{\frac{1}{4}}(x)v(z,\tau)e^{-\int_0^x \frac{b(s)}{2a(s)}ds} + H(z,\tau) \end{aligned} \tag{4.35}$$

where $H(z,\tau) = F(x,t)$.

This leads to

$$\begin{aligned} v_\tau(z,\tau) &= v_{zz}(z,\tau) + \frac{1}{a_0}[c(x) + \frac{a''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(a'(x))^2}{a(x)} + \frac{a'(x)b(x)}{2a(x)} - \\ &\quad \frac{b^2(x)}{4a(x)}v(z,\tau)] + H(x,\tau) \end{aligned}$$

that is,

$$v_t(z,\tau) = v_{zz}(z,\tau) + q(z)v(z,\tau) + H(z,\tau) \tag{4.36}$$

where

$$q(z) = \frac{1}{a_0}\left[c(x) + \frac{a''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(a'(x))^2}{a(x)} + \frac{a'(x)b(x)}{2a(x)} - \frac{b^2(x)}{4a(x)}\right].$$

As for the boundary condition,

$$\alpha u(0,t) + \beta u_x(0,t) = 0$$

it leads to,

$$\alpha_1 v(0,\tau) + \beta_1 v_z(0,\tau) = 0 \tag{4.37}$$

where

$$\begin{cases} \alpha_1 = \{\alpha a(0) + \beta[\frac{1}{4}a'(0) - \frac{b(0)}{2}]\}a^{\frac{-3}{4}}(0) \\ \beta_1 = \beta a_0^{\frac{1}{2}} a^{\frac{-1}{4}}(0) \end{cases} \tag{4.38}$$

Similarly,

$$U(t) = \gamma u(1,t) + \delta u_x(1,t)$$

leads to

$$V(\tau) = \gamma_1 v(1,\tau) + \delta_1 v_z(1,\tau) \tag{4.39}$$

where,

$$\begin{cases} \gamma_1 = \{\gamma a(1) + \delta[\frac{1}{4}a'(1) - \frac{b(1)}{2}]\} a^{\frac{-3}{4}}(1) e^{-\int_0^1 \frac{b(s)}{2a(s)} ds} \\ \delta_1 = \delta \frac{\sqrt{a_0}}{\sqrt{a(1)}} e^{-\int_0^1 \frac{b(s)}{2a(s)} ds}. \end{cases}$$

Hence, (4.32). ■

Applying the operator $1 - \mathbb{K}$ given in Theorem 4 to both sides of the PDE in (4.32), interchanging with $\partial/\partial\tau$ and using the transmutation rule we get,

$$(1-\mathbb{K})\frac{\partial}{\partial\tau} v(z,\tau) = (1-\mathbb{K})[\frac{\partial^2}{\partial z^2} v(z,\tau) + q(z)v(z,\tau) + H(z,\tau)],\ 0 < z < 1,\ \tau > 0$$

$$\frac{\partial}{\partial\tau}(1-\mathbb{K})v = (1-\mathbb{K})(\frac{\partial^2}{\partial z^2} + q(z))v + (1-\mathbb{K})H(z,\tau)$$

$$\frac{\partial}{\partial\tau}(1-\mathbb{K})v = (1-\mathbb{K})\mathbb{L}_0 v + (1-\mathbb{K})H(z,\tau)$$

$$\frac{\partial}{\partial\tau}(1-\mathbb{K})v = \mathbb{L}_1(1-\mathbb{K})v + (1-\mathbb{K})H(z,\tau)$$

$$\frac{\partial}{\partial\tau}\hat{v} = \mathbb{L}_1\hat{v} + \hat{H}(z,\tau)$$

$$\frac{\partial}{\partial\tau}\hat{v} = (\frac{\partial^2}{\partial z^2} + \lambda)\hat{v} + \hat{H}(z,\tau)$$

then, we consider the target system

$$\begin{cases} \hat{v}_\tau(z,\tau) = \hat{v}_{zz}(z,\tau) + \lambda\hat{v}(z,\tau) + \hat{H}(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1\hat{v}(0,\tau) + \beta_1\hat{v}_z(0,\tau) = 0,\ \tau > 0 \\ \gamma_1\hat{v}(1,\tau) + \delta_1\hat{v}_z(1,\tau) = 0,\ \tau > 0 \\ \hat{v}(z,0) = \hat{\psi}(z),\ 0 < z < 1 \end{cases} \tag{4.40}$$

where $\alpha_1^2 + \beta_1^2 = 1$, $\gamma_1^2 + \delta_1^2 = 1$, and $\hat{\psi}$ is a continuous function, whose stability will be established next.

Here, $\hat{v} = (1 - \mathbb{K})v$, $\hat{\psi} = (1 - \mathbb{K})\psi$ and $\hat{H}(z, \tau) = (1 - \mathbb{K})H(z, \tau)$.

The boundary control in this case is obtained as follows.

Since $v = (1 + \mathbb{L})\hat{v}$, that is,

$$v(z, \tau) = \hat{v}(z, \tau) + \int_0^z l(z, y)\hat{v}(y, \tau)dy$$

we have

$$v(z, t) = \hat{v}(1, t) + \int_0^z l(1, y)\hat{v}(y, \tau)dy$$

and

$$v_z(1, \tau) = \hat{v}_z(1, \tau) - l(1, 1)\hat{v}(1, \tau) + \int_0^z l_z(1, y)\hat{v}(y, \tau)dy.$$

Thus,

$$\begin{aligned} V(\tau) \;\; &= \;\; \gamma_1[\hat{v}(1, \tau) + \int_0^1 l(1, y)\hat{v}(y, \tau)dy] + \delta_1[\hat{v}_z(1, \tau) + \\ & \qquad l(1, 1)\hat{v}(1, \tau) + \int_0^1 l_z(1, y)\hat{v}(y, \tau)dy] \end{aligned}$$

that is,

$$V(\tau) = \delta_1 l(1, 1)\hat{v}(1, \tau) + \int_0^1 [\gamma_1 l(1, y) + \delta_1 l_z(1, y)]\hat{v}(y, \tau)dy. \tag{4.41}$$

We shall prove the stability of the target system (4.40)

Let $\{\omega_n^2, \varphi_n\}$, be the (simple) eigenvalues and corresponding eigenfunctions of the regular Sturm-Liouville problem

$$\begin{cases} \frac{d^2}{dz^2}\varphi(z) + \omega^2\varphi(z) = 0, \ z \in (0,1) \\ \alpha_1\varphi(0) + \beta_1\varphi_z(0) = 0 \\ \gamma_1\varphi(0) + \delta_1\varphi_x(1) = 0 \end{cases}$$

where $\alpha_1^2 + \beta_1^2 = 1$, $\gamma_1^2 + \delta_1^2 = 1$

These eigenvalues can be ordered as $\omega_1^2 < \omega_2^2 < ...... \uparrow \infty$ as $n \to \infty$.

Let

$$\hat{H}(z,\tau) = \sum_{n\geq 1} h_n(\tau)\varphi_n(z)$$

and

$$\psi(z) = \sum_{n\geq 1} c_n\varphi_n(z)$$

be the eigenfunctions expansions of H and $\hat{\psi}$ with respect to the basis $\{\varphi_n\}_{n\geq 1}$ and L^2 inner product $\langle . \rangle$. We have,

$$h_n(\tau) = \left\langle \hat{H}(z,\tau), \varphi_n(z) \right\rangle / \left\|\varphi_n\right\|^2$$

and

$$c_n = \langle \psi, \varphi_n \rangle / \left\|\varphi_n\right\|^2$$

for $n \geq 1$.

We shall seek the solution of (4.40) in the form

$$\hat{v}(z,\tau) = \sum_{n\geq 1} a_n(\tau)\varphi_n(z).$$

This gives the infinite system of equations

$$\begin{cases} \frac{da_n}{d\tau} = (-\omega_n^2 + \lambda)a_n(\tau) + h_n(\tau) \\ a_n(0) = c_n \end{cases}$$

for $n \geq 1$.

Thus,

$$a_n(\tau) = e^{-(\omega_n^2-\lambda)\tau}c_n + \int_0^\tau e^{-(\omega_n^2-\lambda)(\tau-\xi)}h(\xi)d\xi$$

and

$$\hat{v}(z,\tau) = \sum_{n\geq 1} c_n e^{-(\omega_n^2-\lambda)\tau}\varphi_n(z) + \sum_{n\geq 1}\varphi_n(z)\int_0^\tau e^{-(\omega_n^2-\lambda)(\tau-\xi)}h(\xi)d\xi.$$

Now,

$$\begin{aligned} |h_n(\tau)| &= \frac{\left|\int_0^1 \hat{H}(z,\tau)\varphi_n(z)dz\right|}{\|\varphi_n\|^2} \\ &\leq \frac{\int_0^1 \left|\hat{H}(z,\tau)\right| |\varphi_n(z)|\, dz}{\|\varphi_n\|^2} \\ &\leq A_0 e^{-\rho\tau}\frac{\|\varphi_n(z)\|}{\|\varphi_n\|^2} = \frac{A_0 e^{-\rho\tau}}{\|\varphi_n\|}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int_0^\tau e^{-(\omega_n^2-\lambda)(\tau-\xi)}h(\xi)d\xi\right| &\leq e^{-(\omega_n^2-\lambda)\tau}\int_0^\tau e^{(\omega_n^2-\lambda)\xi}\frac{A_0 e^{-\rho\xi}}{\|\varphi_n\|}d\xi \\
&= e^{-(\omega_n^2-\lambda)\tau}\frac{A_0}{\|\varphi_n\|}\frac{e^{(\omega_n^2-\lambda-\rho)\tau}-1}{\omega_n^2-\lambda-\rho} \\
&= \frac{A_0}{\|\varphi_n\|}\frac{e^{-\rho\tau}-e^{-(\omega_n^2-\lambda)\tau}}{\omega_n^2-\lambda-\rho}.
\end{aligned}$$

If

$$\hat{\rho} = \inf\{\rho, \omega_1^2 - \lambda\}$$

then

$$|\hat{v}(z,\tau)| \leq A_1 e^{-\hat{\rho}\tau}.$$

Thus, we have proved the following,

Theorem 8 *For the system (4.40) if $\hat{H}(z,\tau)$ is such that*

$$\left|\hat{H}(z,\tau)\right| \leq A_0 e^{-\rho\tau}$$

for $z \in (0,1)$, and $\tau > 0$, for some ρ and $\hat{\rho} = \inf\{\rho, \omega_1^2 - \lambda\} > 0$ then $\hat{v}(z,\tau)$ satisfies the estimate

$$|\hat{v}(z,\tau)| \leq A_1 e^{-\hat{\rho}\tau}.$$

for all $\tau > 0$ where A_0 and A_1 are absolute constants, that is the system is exponentially stable.

Combining the above results, we get the following,

Theorem 9 *The transmutation operator* $1 - \mathbb{K}$ *maps (4.32) into the exponentially stable (4.40) and the boundary control* V *is given by (4.41) .*

4.3 Time & space dependent coefficients

Consider now, the PDE system,

$$\begin{cases} u_t(x,t) = a(x,t)u_{xx}(x,t) + b(x,t)u_x(x,t) + c(x,t)u(x,t),\ 0 < x < 1, t > 0 \\ \alpha u(0,t) + \beta u_x(0,t) = 0, t > 0 \\ \gamma u(1,t) + \delta u_x(1,t) = U(t), t > 0 \\ u(x,0) = \phi(x),\ 0 < x < 1 \end{cases} \tag{4.42}$$

Let $a(x,t) = a_0(x) + a_1(x,t), b(x,t) = b_0(x) + b_1(x,t)$, and $c(x,t) = c_0(x) + c_1(x,t)$.

We may take $a_0(x) = a(x,0)$, $b_0(x) = b(x,0)$, $c_0(x) = c(x,0)$ and we assume ϕ, $c_0 \in C[0,1]$, $a_0 > 0$, $a_0 \in C^2[0,1]$, $b_0 \in C^1[0,1]$ and U is the boundary control.

The PDE is rewritten as,

$$u_t(x,t) = [a_0(x) + a_1(x,t)]u_{xx}(x,t) + [b_0(x) + b_1(x,t)]u_x(x,t) + [c_0(x) + c_1(x,t)]u(x,t)$$

thus,

$$\begin{aligned} u_t(x,t) &= a_0(x)u_{xx}(x,t) + b_0(x)u_x(x,t) + c_0(x)u(x,t) + \\ &\quad a_1(x,t)u_{xx}(x,t) + b_1(x,t)u_x(x,t) + c_1(x,t)u(x,t) \end{aligned} \tag{4.43}$$

Theorem 10 *Let $U^{[0]}$ be the boundary control for the system*

$$\left\{\begin{array}{c} u_t^{[0]}(x,t) = a_0(x)u_{xx}^{[0]}(x,t) + b_0(x)u_x^{[0]}(x,t) + c_0(x)u^{[0]}(x,t), 0<x<1,\ t>0 \\ \alpha u^{[0]}(0,t) + \beta u_x^{[0]}(0,t) = 0,\ t>0 \\ \gamma u^{[0]}(1,t) + \delta u_x^{[0]}(1,t) = U^{[0]}(t)\ ,\ t>0 \\ u^{[0]}(x,0) = \phi(x),\ 0<x<1 \end{array}\right. \tag{4.44}$$

and $u^{[0]}$ the corresponding solution, and for $k \geq 1$, let be the boundary control $U^{[k]}$ for the system

$$\left\{\begin{array}{c} u_t^{[k]}(x,t) = a_0(x)u_{xx}^{[k]}(x,t) + b_0(x)u_x^{[k]}(x,t) + c_0(x)u^{[k]}(x,t) + F^{[k]}(x,t),\ 0<x<1,\ t>0 \\ \alpha u^{[k]}(0,t) + \beta u_x^{[k]}(0,t) = 0,\ t>0 \\ \gamma u^{[k]}(1,t) + \delta u_x^{[k]}(1,t) = U^{[k]}(t),\ t>0 \\ u^{[k]}(x,0) = 0,,\ 0<x<1 \end{array}\right. \tag{4.45}$$

and $u^{[k]}$ the corresponding solution, where,

$$F^{[k]}(x,t) = a_1(x,t)u_{xx}^{[k-1]}(x,t) + b_1(x,t)u_x^{[k-1]}(x,t) + c_1(x,t)u^{[k-1]}(x,t). \tag{4.46}$$

Then, $U(t) = \sum_{k=0}^{\infty} U^{[k]}(t)$ is the boundary control for the original system (4.42) and $u(x,t) = \sum_{k=0}^{\infty} u^{[k]}(x,t)$ is its corresponding solution.

Proof. This is a direct consequence of the decomposition (4.42) and the use of successive approximation:

Problem (4.44) is solved using the approach presented in section (4.1), while problem (4.45) is solved using approach presented is section (4.2) and therefore the boundary control is

$$U(t) = \sum_{k\geq 0} U^{[k]}(t)$$

and the corresponding solution of (4.42) is

$$u(x,t) = \sum_{k\geq 0} u^{[k]}(x,t).$$

Each of the system (4.44) and (4.45) is first transformed using the change of variables given in (4.2), to obtain,

$$\begin{cases} v_\tau^{[0]}(z,\tau) = v_{zz}^{[0]}(z,\tau) + q(z)v^{[0]}(z,\tau),\ 0 < z < 1,\ \tau > 0 \\ \alpha_1 v^{[0]}(0,\tau) + \beta_1 v_z^{[0]}(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 v^{[0]}(1,\tau) + \delta_1 v_z^{[0]}(1,\tau) = V^{[0]}(\tau),\ \tau > 0 \\ v^{[0]}(z,0) = \psi(z),\ 0 < z < 1 \end{cases}$$

and

$$\begin{cases} v_\tau^{[k]}(z,\tau) = v_{zz}^{[k]}(z,\tau) + q(z)v^{[k]}(z,\tau) + H^{[k]}(z,\tau), 0 < z < 1,\ \tau > 0 \\ \alpha_1 v^{[k]}(0,\tau) + \beta_1 v_z^{[k]}(0,\tau) = 0,\ \tau > 0 \\ \gamma_1 v^{[k]}(1,\tau) + \delta_1 v_z^{[k]}(1,\tau) = V^{[k]}(\tau),\ \tau > 0 \\ v^{[k]}(z,0) = 0,\ 0 < z < 1 \end{cases}$$

respectively, which then are mapped into the target systems using the same transmutation $1-\mathbb{K}$

$$\begin{cases} \hat{v}^{[0]}_{\tau}(z,\tau) = \hat{v}^{[0]}_{zz}(z,\tau) + q(z)\hat{v}^{[0]}(z,\tau) \\ \alpha_1\hat{v}^{[0]}(0,\tau) + \beta\hat{v}^{[0]}_{z}(0,\tau) = 0 \\ \gamma_1\hat{v}^{[0]}(1,\tau) + \delta\hat{v}^{[0]}_{z}(1,\tau) = 0 \\ \hat{v}^{[0]}(z,0) = \hat{\psi}(z) \end{cases}$$

and

$$\begin{cases} \hat{v}^{[k]}_{\tau}(z,\tau) = \hat{v}^{[k]}_{zz}(z,\tau) + q(z)\hat{v}^{[k]}(z,\tau) + \hat{H}^{[k]}(z,\tau) \\ \alpha_1\hat{v}^{[k]}(0,\tau) + \beta_1\hat{v}^{[k]}_{z}(0,\tau) = 0 \\ \gamma_1\hat{v}^{[k]}(1,\tau) + \delta_1\hat{v}^{[k]}_{z}(1,\tau) = 0 \\ \hat{v}^{[k]}(z,0) = 0 \end{cases}$$

for $k \geq 1$, respectively, leading to their solutions. ■

4.4 *Examples*

We shall work out a few examples illustrating the effectiveness of the method presented. We shall consider the boundary control problems given by

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + q(x)u(x,t),\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ t > 0 \\ u(1,t) = U(t),\ t > 0 \\ u(x,0) = u_0(x),\ 0 < x < 1 \end{cases}$$

with different functions q, $u0$ and λ appearing in the target system below,

$$
\begin{cases}
\widehat{u}_t(x,t) = \widehat{u}_{xx}(x,t) + \lambda \widehat{u}(x,t),\ \ 0 < x < 1,\ \ t > 0 \\
\widehat{u}(0,t) = 0,\ \ t > 0 \\
\widehat{u}(1,t) = 0,\ \ t > 0 \\
\widehat{u}(x,0) = \widehat{u}_0(x),\ \ 0 < x < 1
\end{cases}
$$

where $\widehat{u} = (\mathbb{I} - \mathbb{K})u$, $u = (\mathbb{I} - \mathbb{L})\ \widehat{u}$, $\widehat{u}_0 = (\mathbb{I} - \mathbb{K})u_0$. $\mathbb{K}$ and $\mathbb{L}$ are Volterra operators with kernels k and l respectively. In each example we shall plot the initial condition u_0 of the given system, the solution u of the free system (i.e., uncontrolled $U(t) = 0$), the kernels k and l of the transmutation and its inverse, the solution $\widehat{u}$ of the target system and its initial condition $\widehat{u}_0$, and finaly, the boundary control U and the solution u of the given system. In each case we have truncated the series expansions of the kernel functions to 10 terms and taken 10 subdivision in the x-interval, while taking 15 points over the t-interval $[0, 10]$.

Obviously, while both free systems were unstable, the use of the boundary control obtained stablilize the given systems. In the third example, three iterates of the boundary control were enough to stabilize the system. The t-interval was taken as $[0, 20]$.

Example 1 $q(x) = 100x$, $u_0(x) = x(1-x)$, $\lambda = 3$

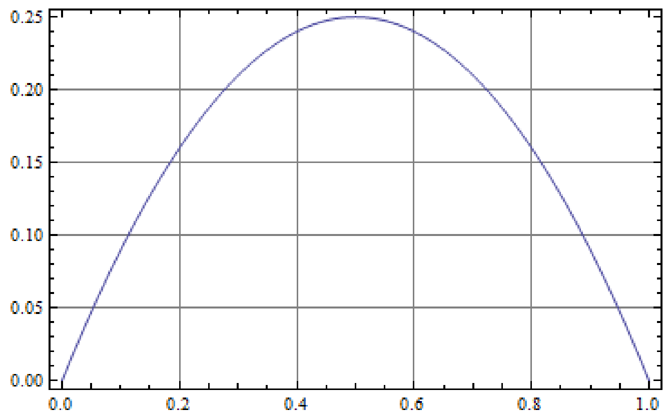

Figure 4-1: Initial condition u_0 for Example1

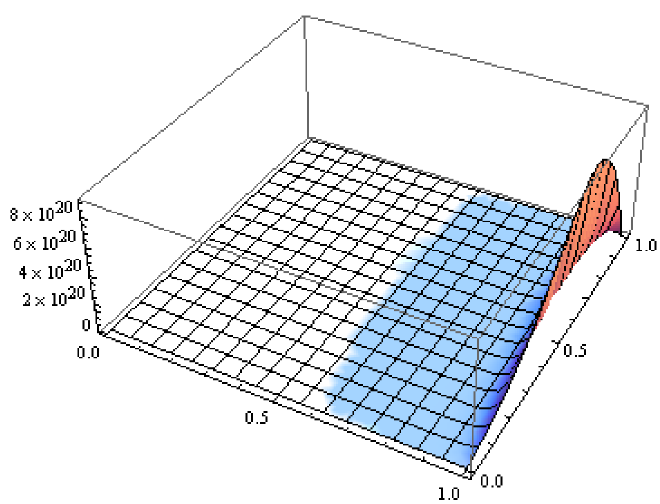

Figure 4-2: Solution of the uncontrolled system for Example 1

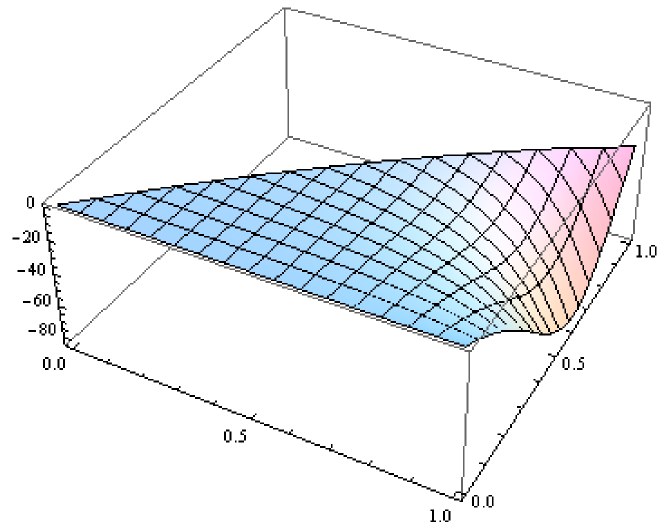

Figure 4-3: k kernel for Example 1

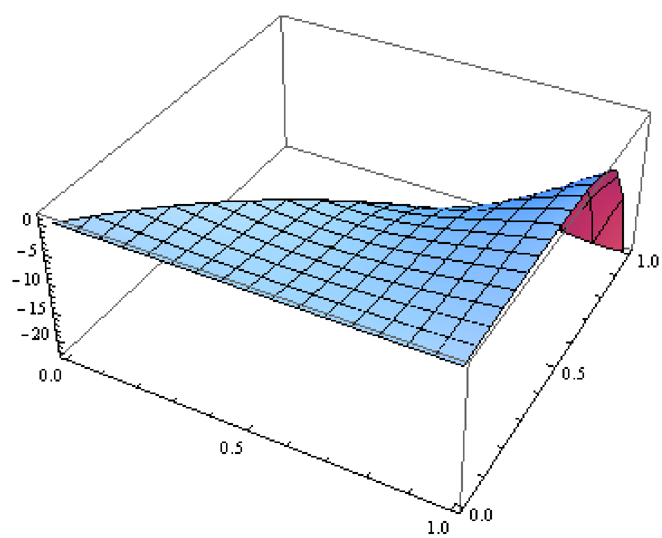

Figure 4-4: l kernel for Example 1

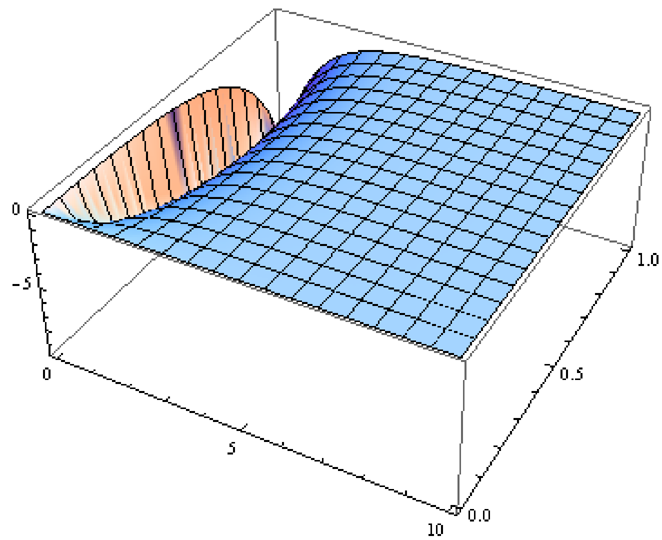

Figure 4-5: Solution $\widehat{u}$ of the Target System for Example 1

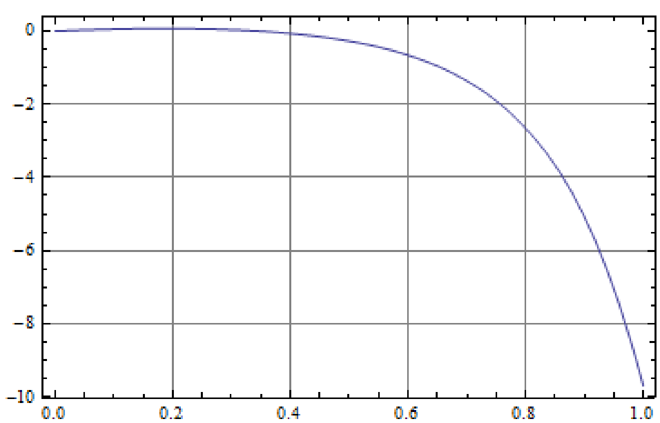

Figure 4-6: Initial condition $\widehat{u}_0$ for the Target System in Example 1

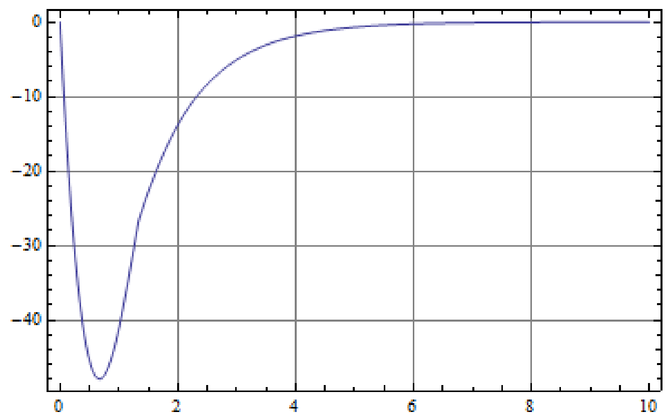

Figure 4-7: Boundary control U for the system in Example 1

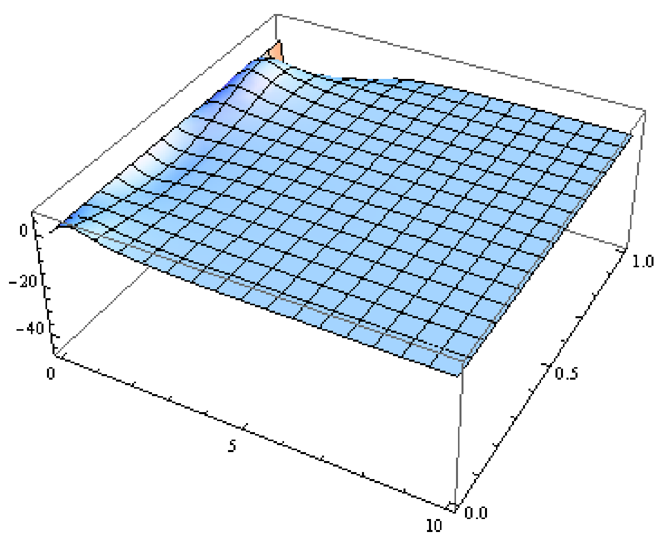

Figure 4-8: Solution u of the original system in Example 1

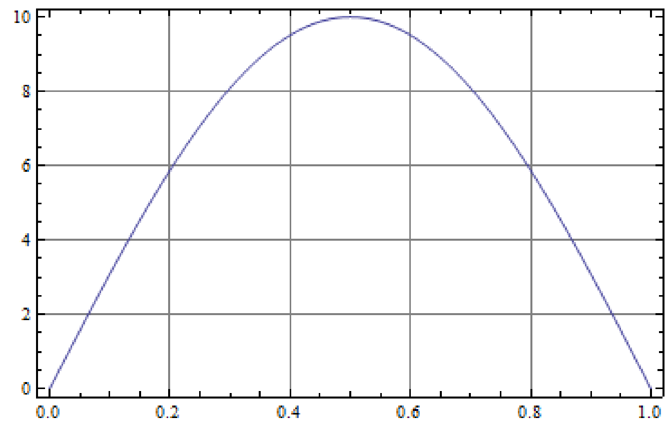

Figure 4-9: Initial condition u_0 for Example2

Example 2 $q(x) = 100x^2$, $u_0(x) = 10 \sin \pi x$, $\lambda = 3$

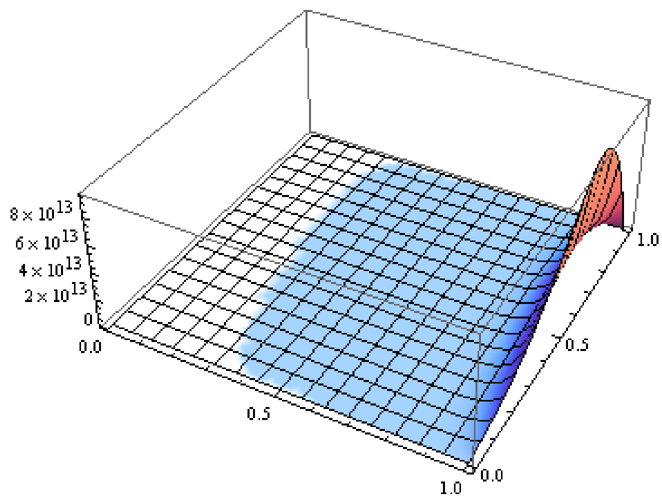

Figure 4-10: Solution of the uncontrolled system for Example 2

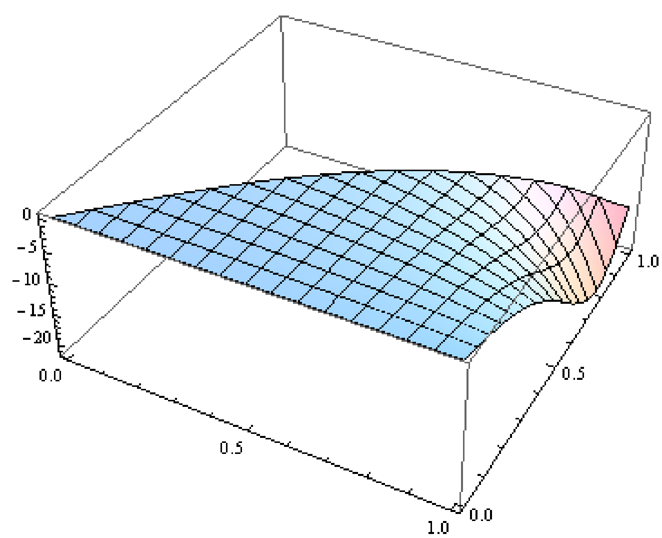

Figure 4-11: k kernel for Example 2

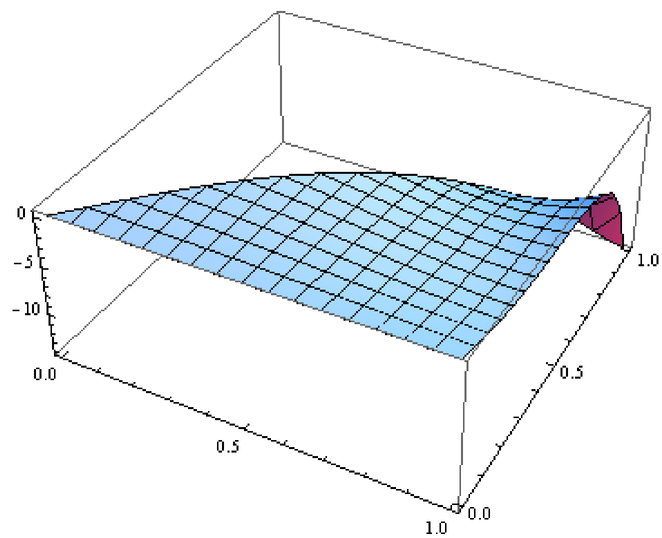

Figure 4-12: l kernel for Example 2

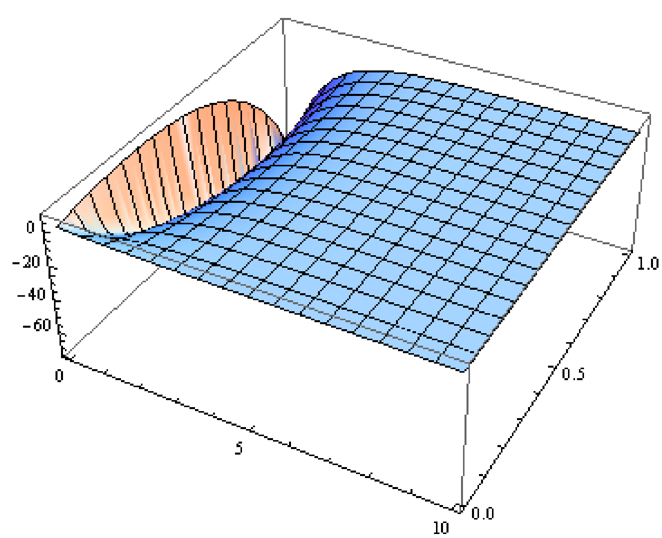

Figure 4-13: Solution $\widehat{u}$ of the Target System for Example 2

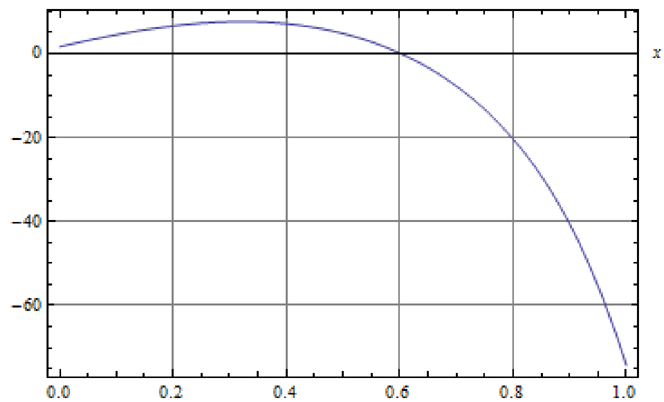

Figure 4-14: Initial condition $\widehat{u}_0$ for the Target System in Example 2

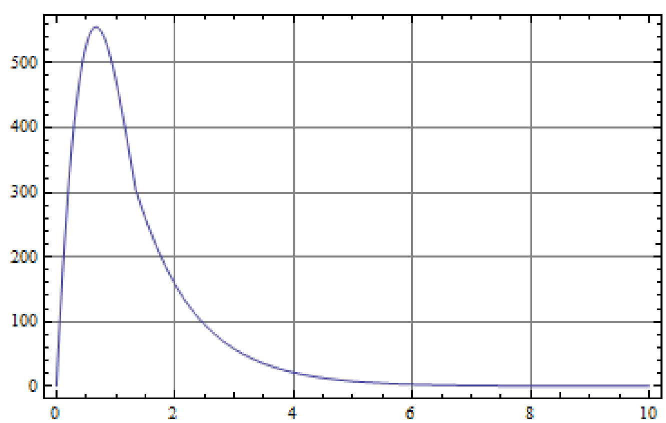

Figure 4-15: Boundary control U for the system in Example 2

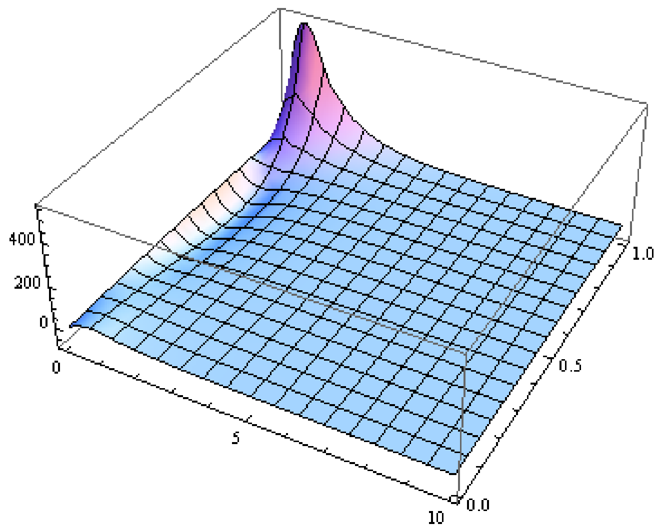

Figure 4-16: Solution u of the original system in Example 2

Example 3 $q(x) = 100x + xt$, $u_0(x) = 10 \sin \pi x$, $\lambda = 5$

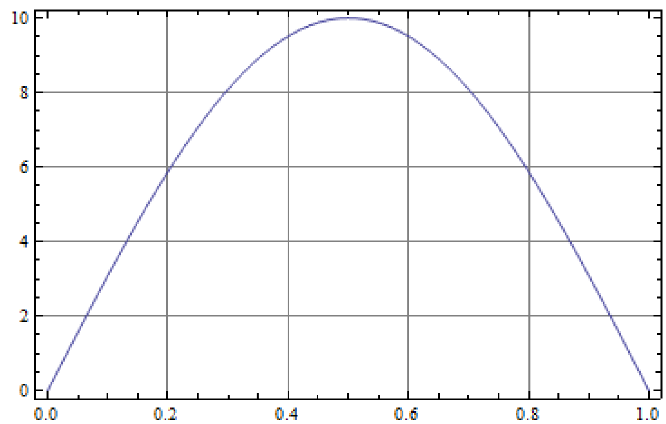

Figure 4-17: Initial Condition u_0 for Example 3

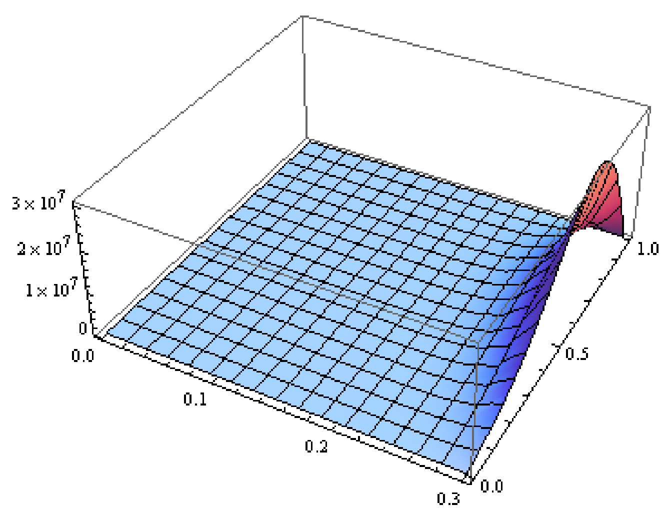

Figure 4-18: Solution of the uncontrolled system for Example 3

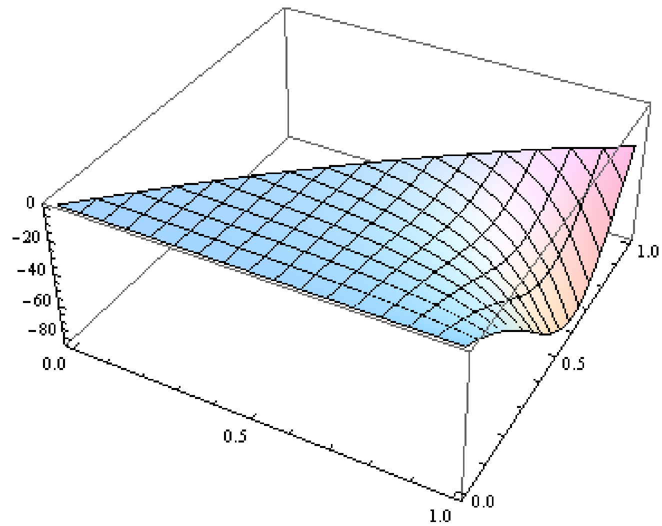

Figure 4-19: $k-$ kernel for Example 3

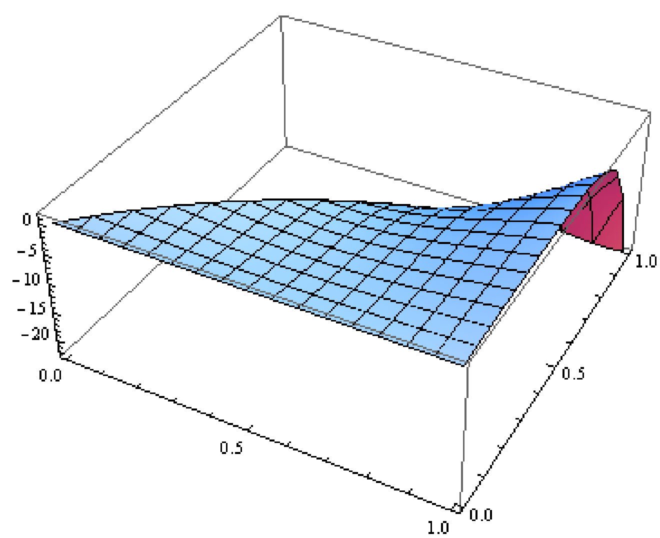

Figure 4-20: $l-$ kernel for Example 3

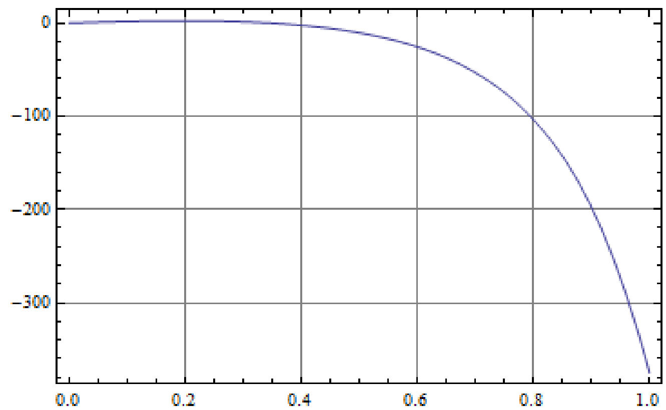

Figure 4-21: Initial condition $\widehat{u}_0$ for the Target System in Example 3

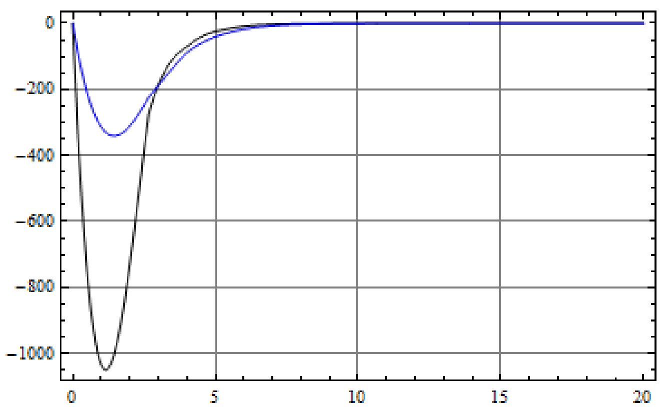

Figure 4-22: Boundary Control $U = U^{[0]}$and $U = U^{[0]} + U^{[1]}$

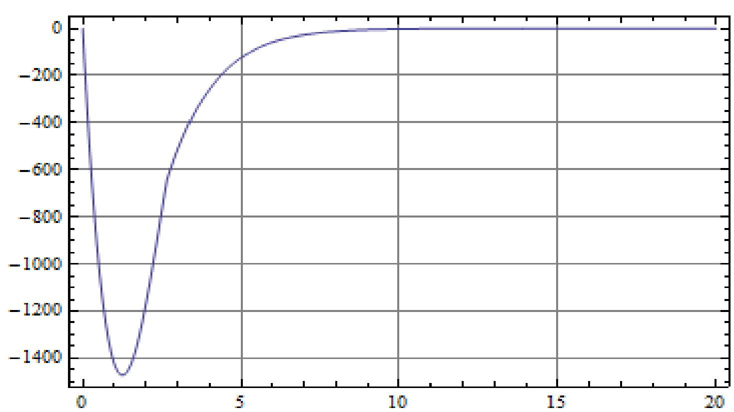

Figure 4-23: Boundary Control $U = U^{[0]} + U^{[1]} + U^{[2]}$ in Example 3

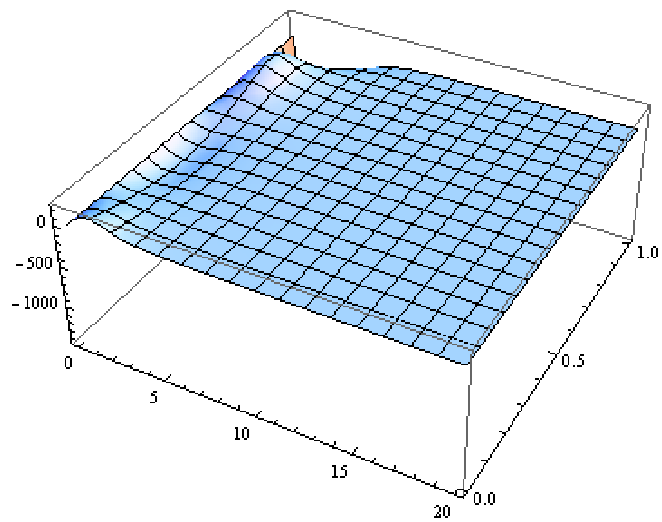

Figure 4-24: Solution u of the original system in Example 3

Chapter 5

CONCLUSION

We have considered the linear heat equation with space and time dependent coefficients with Robin boundary condition at $x = 0$ and Robin boundary control at $x = 1$.

Without control, the original system is unstable if certain function coefficient is positive and large. After normalization, the PDE system is mapped into a stable target system using a bounded invertible linear transformation (transmutation) well known in spectral and scattering theories. Under suitable conditions the exponential stability of the system with prescribed decay rate is achieved. This decay rate is improved if we have some information on the initial condition. We have worked out three examples with space and time dependent coefficients.

As future work, one can try to stabilize the nonlinear problem

$$u_t(x,t) = u_{xx}(x,t) + q(x,t,u(x,t))u(x,t)$$

where q depends not only on x, t but also on u itself.

Bibliography

[1] O.-M.Aamo and M.Krstic,Flow Control by Feedback:Stabilization and Mixing, Springer, New York, 2003.

[2] H. T. Banks, R. C. Smith, and Y. Wang, Smart Material Structures: Modeling, Estimation, and Control, Wiley, New York, Masson, Paris, 1996.

[3] A. Bensoussan, G. Da Prato, M. C. Delfour, and S.K. Mitter, Representation and Control of Infinite-dimensional Systems, 2nd ed., Birkhauser, Boston, 2006.

[4] J.Bentsman and Y . Orlov,"Reduced spatial order model reference adaptive control of spatially varying distributed parameter systems of parabolic and hyperbolic types," Int. J. Adapt. Control Signal Process. vol. 15, pp. 679-696, 2001.

[5] M. Bohm, M. A. Demetriou, S. Reich, and I. G. Rosen, "Model reference adaptive control of distributed parameter systems, " SIAM J. Control Optim., Vol. 36, No. 1, pp. 33-81, 1998.

[6] P. Christofides, Nonlinear and Robust Control of Partial Differential Equation System: Methods and Applications to Transport-Reaction Processes, Birkhauser, Boston, 2001.

[7] R. F. Curtain and H. J. Zwart, An Introduction to Infinite Dimensional linear System Theory, Springer-Verlag, New York, 1995.

[8] M. S. de Queiroz, D. M. Dawson, M. Agarwal, and F. Zhang, “Adaptive nonlinear boundary control of a flexible link robot arm,” IEEE Trans-actions on Robotics and Automation, vol. 15, no. 4, pp. 779–787, 1999.

[9] M. A. Demetriou and K. Ito, “Optimal on-line parameter estimation for a class of infinite dimensional systems using Kalman filters,” Proceedings of the American Control Conference, 2003.

[10] T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, “Adaptive boundary and point control of linear stochastic distributed parameter systems,” SIAM J. Control Optim., vol. 32, no. 3, pp. 648-672, 1994.

[11] M. D. Gunzburger, Perspectives in Flow Control and Optimization, SALM, Philadelphia, 2002.

[12] M. S. dr Queiroz, D. M. Dawson, S. P. Nagarkatti, and F. Zhang, Lyapunov-Based Control of Mechanical Systems, Birkhauser, Basel,2000.

[13] K. S. Hong and J. Bentsman, “Direct adaptive control of parabolic systems: Algorithm synthesis, and convergence, and stability analysis,” IEEE Trans. Automatic Control, vol. 39, pp. 2018-2033, 1994.

[14] M. Jovanovic and B. Bamieh, “Lyapunov-based distributed control of systems on lattices,” IEEE Transactions on Automatic Control, vol. 50, pp. 422–433, 2005.

[15] J. Jia,“Boundary Feedback Control of Unstable Heat Equation with Space and Time Dependent Coefficient” axXiv: math/0610253v1 [math.OC] 7 Oct 2007.

[16] T. Kobayashi, “Adaptive stabilization of the Kuramoto-Sivashinsky equation,” International Journal of Systems Science, vol. 33, pp. 175–180, 2002.

[17] T. Kobayashi, “Adaptive regulator design of a viscous Burgers’ system by boundary control,” IMA Journal of Mathematical Control and Information, vol. 18, pp. 427–437, 2001.

[18] M.Krstic and A. Smyshlyaev, Andrey, Boundary Control of PDEs: A Course on Backstepping Designs, Advances in Design and Control, Siam,2008.

[19] M. Krstic and A. Smyshlyaev, “Adaptive Boundary Control for Unstable Parabolic PDEs—Part I: Lyapunov Design,” submitted to IEEE Transactions on Automatic Control, vol.53,no.7,August 2008.

[20] J. E. Lagenese, Boundary Stabilization of Thin Plates, SLAM,Philadelphia, 1989.

[21] I. Lasiecka, Mathematical Control Theory of Coupled PDEs, SLAM, Philadelphia 2002.

[22] I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equation: Continuous and Approximation Theories, 2 vols., Cambridge University Press, Cambridge, UK,2000.

[23] Lions, J.L. Contrôle Optimal des Systèmes Gouvernés par des Équations aux Dérivées Partielles, 1968, Dunod, Paris, Gauthier-Villars.

[24] W. Liu and M. Krstic, "Adaptive control of Burgers' equation with unknown viscosity," International Journal of Adaptive Control and Signal Processing, vol. 15, pp. 745–766, 2001.

[25] H. Logemann and S. Townley, "Adaptive stabilization without identification for distributed parameter systems: An overview," IMA J. Math. Control and Information, vol. 14, pp. 175-206, 1997.

[26] Z. H. Luo, B. Z. Guo, and O. Morgul, Stability and Stabilization of Infinte Dimensional Systems with Applications, Springer-Verlag, New York, 1999.

[27] Y. Orlov, "Sliding mode observer-based synthesis of state derivative-free model reference adaptive control of distributed parameter systems," J. of Dynamic Systems, Measurement, and Control, vol. 122, pp. 726-731, 2000.

[28] A. Smyshlyaev and M. Krstic, "Adaptive boundary control for unstable parabolic PDEs—Part II: Estimation-based designs," accepted 15 February 2007.

[29] A. Smyshlyaev and M. Krstic, "Adaptive boundary control for unstable parabolic PDEs—Part III: Output-feedback examples with swapping identifiers," accepted 15 February 2007.

[30] A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of partial integro-differential equations," IEEE Trans. on Automatic Control, Vol. 49, No. 12, pp. 2185-2202, 2004.

[31] V. Solo and B. Bamieh, "Adaptive distributed control of a parabolic system with spatially varying parameters," Proc. 38th IEEE Conf. Decision and Control, pp. 2892–2895, 1999.

[32] S. Townley, "Simple adaptive stabilization of output feedback stabilizable distributed parameter systems," Dynamics and Control, vol. 5 pp. 107–123, 1995.

[33] J. T.-Y. Wen and M. J. Balas, "Robust adaptive control in Hilbert space," Journal of Mathematical Analysis and Applications, vol. 143, pp. 1–26, 1989.

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